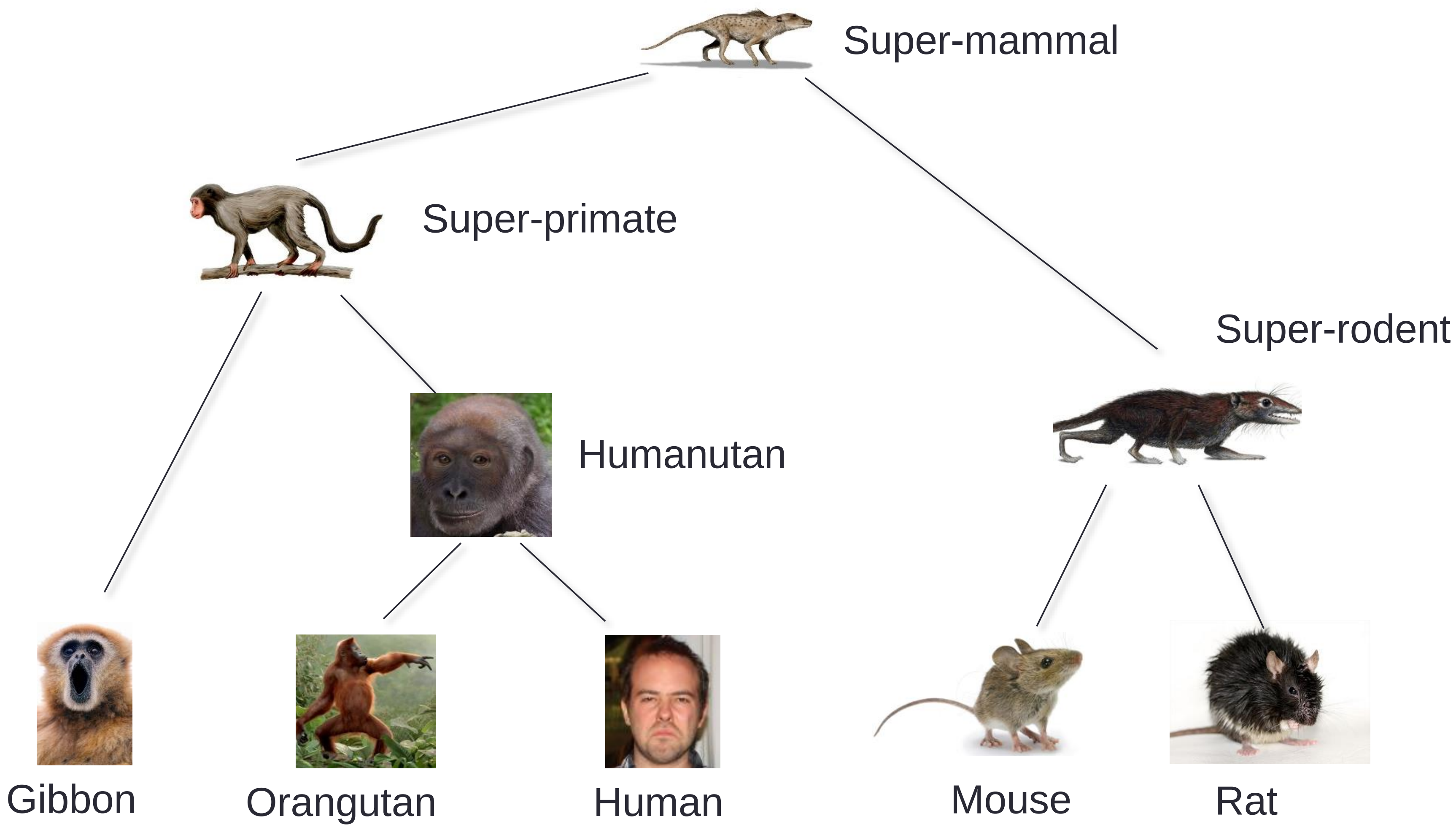
ON STRONGLY CHORDAL GRAPHS THAT ARE NOT LEAF POWERS

Manuel Lafond (University of Ottawa)

Motivation

Species evolve in a tree-like manner.



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Species evolve in a tree-like manner.
We can only observe the species that exist today, which are **the leaves of the tree**.

To infer the tree, we compare DNA sequences.


Gibbon


Orangutan


Human


Mouse


Rat

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To infer the tree, we compare DNA sequences => Distance matrix D

	G	O	H	M	R
G	0	0.33	0.25	0.52	0.61
O		0	0.31	0.61	0.36
H			0	0.55	0.56
M				0	0.2
R					0

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Let's say
$D_{xy} \leq 0.5$ means "close" (1)
$D_{xy} > 0.5$ means "distant" (0)


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Species evolve in a tree-like manner.
We can only observe the species that exist today, which are **the leaves of the tree**.

To infer the tree, we compare DNA sequences => Distance matrix D

	G	O	H	M	R
G		1	1	0	0
O			1	0	1
H				0	0
M					1
R					

Let's say
$D_{xy} \leq 0.5$ means "close" (1)
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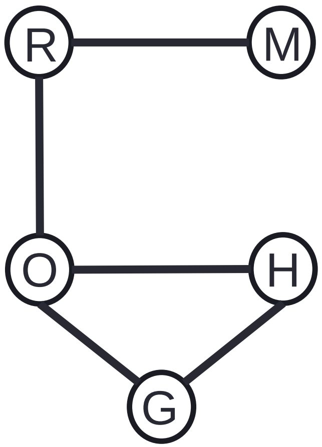
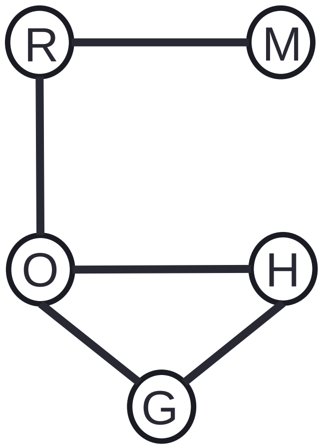


Motivation

Does this graph make any sense, « biologically » speaking?

Is there a tree with leafset {R,M,O,H,G} in which
the pairs that share an edge are **closer than the pairs that don't ?**

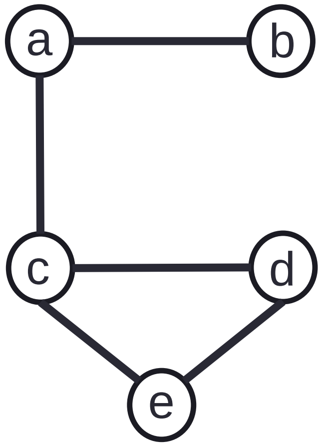


k-leaf power

A graph G is a **k-leaf power** if there exists a tree T such that:

- **L(T) = V(G),** where L(T) is the set of leaves of T
- **$uv \in E(G) \Leftrightarrow d_T(u, v) \leq k$**

G:



Is G is 3-leaf power?

k-leaf power

A graph G is a **k-leaf power** if there exists a tree T such that:

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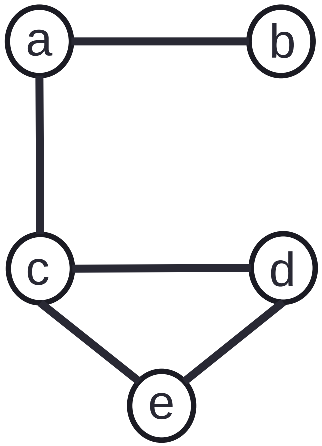
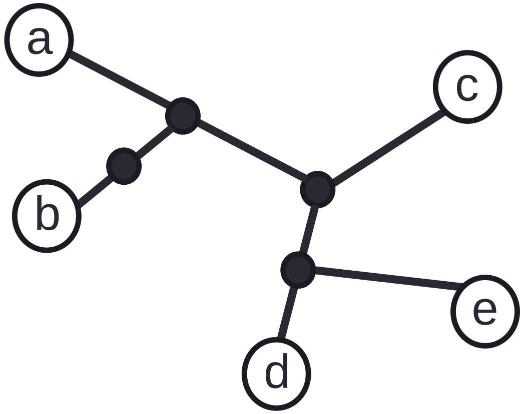
G:



T:



d(a, b) ≤ 3
d(a, c) ≤ 3
d(b, c) > 3
…

Is G is 3-leaf power? **Yes!**

T is called a k-leaf root of G
(or a **leaf root** for short)

k-leaf power

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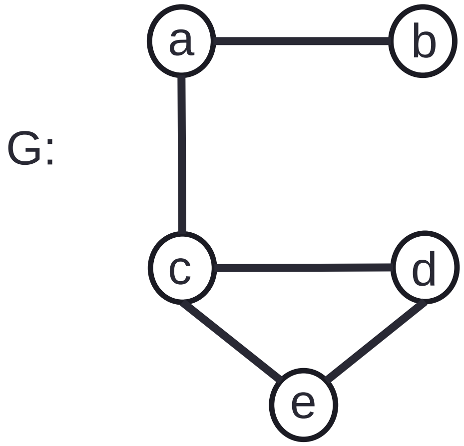

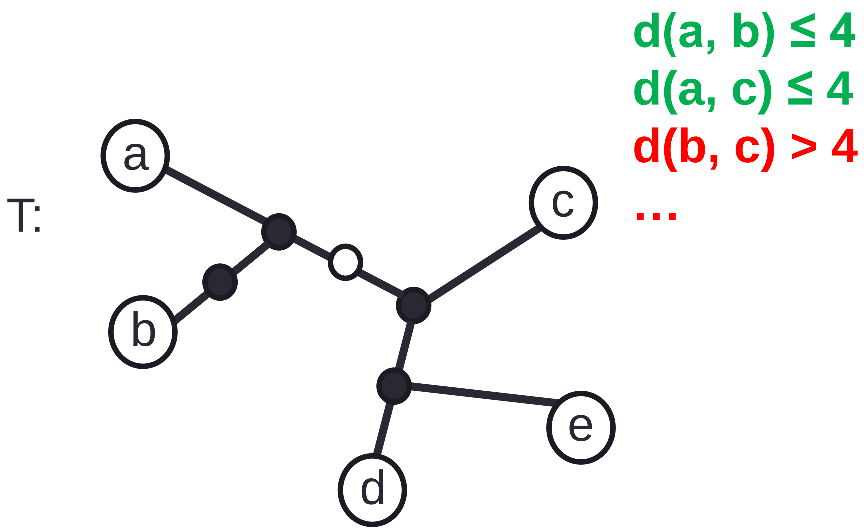

Is G is 3-leaf power? **Yes!**
Is G a 4-leaf power? **Yes!**

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k-leaf power

A graph G is a **k-leaf power** if there exists a tree T such that:

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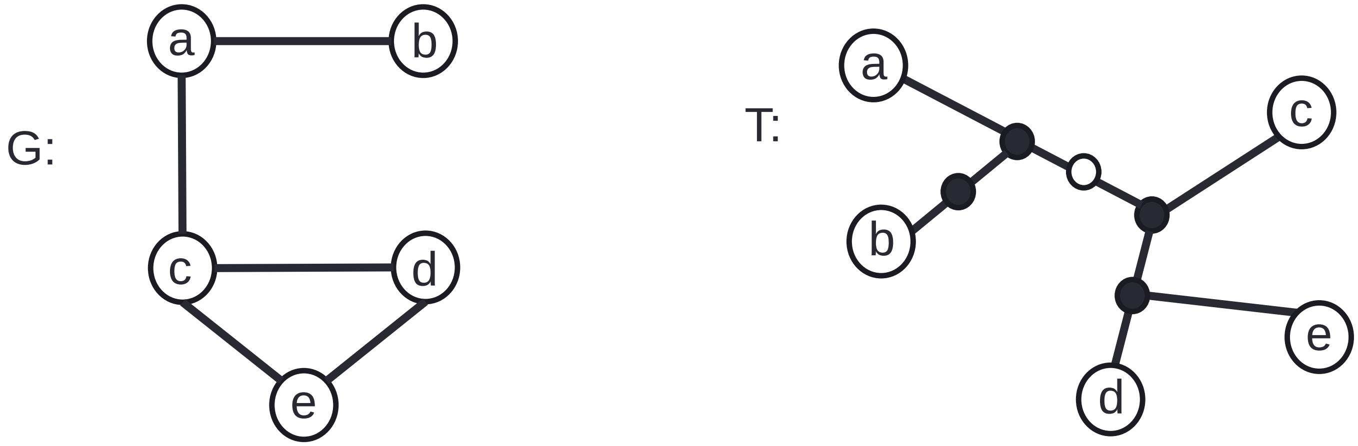
- **uv ∈ E(G) ⇔ d_T(u, v) ≤ k**



Is G is 3-leaf power? **Yes!**

Is G a 4-leaf power? **Yes!**

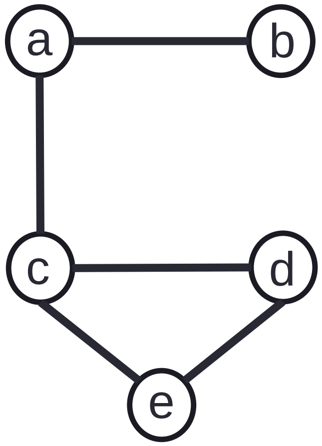
Is G a 2-leaf power? No… (because 2-leaf powers are the P_3-free graphs)

k-leaf power

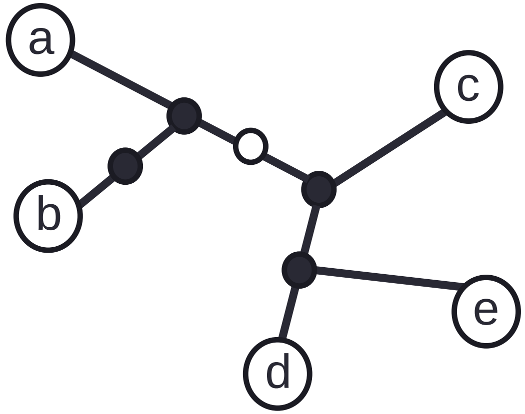
Why the name ‘leaf power’ ?

Take the k-th power of T, keep only the leaves, the result is G.

G:

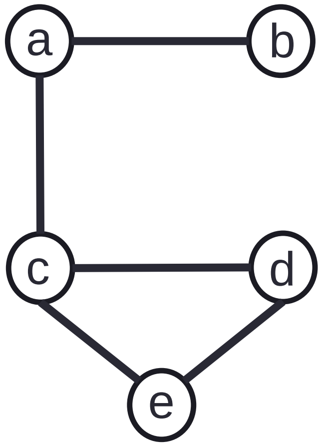


T:

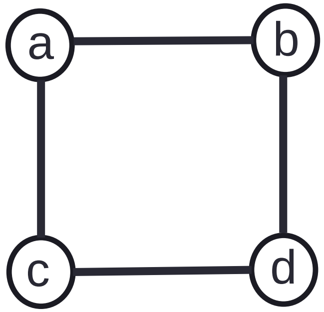


Leaf power

A graph G is a **leaf power** if it is a k-leaf power **for some k**.



Leaf power



Not leaf power

The problems

Graph theoretical perspective

- Characterize the class of k-leaf powers, for every k.
- Characterize the class of leaf powers.

Algorithmic perspective

Given a graph G, decide:

- whether G is a leaf power.
- whether G is a k-leaf power, where k is given.
- whether G is a k-leaf power, where k is fixed.

What is known?

Graph theoretical perspective

- Characterize the class of k-leaf powers, for every k.
  - 2-leaf powers are the P3-free graphs, 3-leaf powers are the chordal (bull,dart,gem)-free graphs
  - 4-leaf powers and 5-leaf powers also have a chordality + forbidden subgraph characterization (Rautenbach, 2006, Brandstädt & Sritharan, 2008)
  - Open for $k \geq 6$

What is known?

Graph theoretical perspective

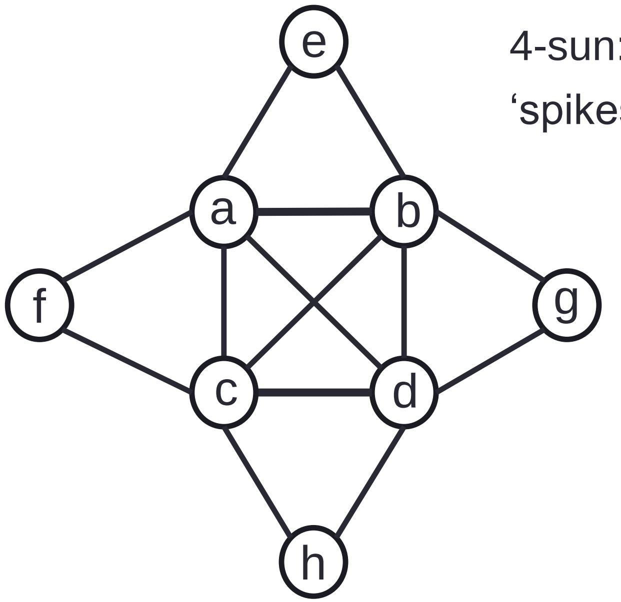
- Characterize the class of leaf powers.
  - Leaf powers are **strongly chordal** (chordal + sun-free)
  - Some subclasses of strongly chordal graphs are known to be leaf powers (ptolemaic, interval, rooted directed path, strictly chordal)
  - (Brandstädt, Hundt, Mancini, Wagner, Kennedy, Lin, Yan, 2010 +/- a few years)
  - Only **7 strongly chordal graphs are known to not be leaf powers** (Nevries and Rosenke, 2015)

  - **Conjecture:** a graph G is a leaf power iff it is strongly chordal and does not contain one of these 7 graphs (as an induced subgraph).

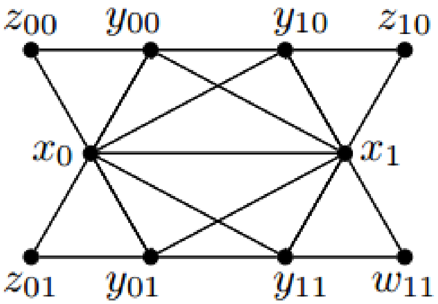
Strongly chordal graphs

A graph is chordal is every cycle on at least 4 vertices has a chord.

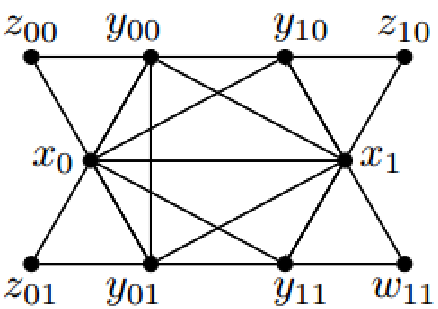
A graph is strongly chordal if it is chordal and sun-free.



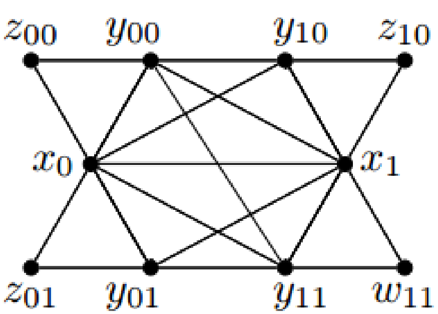
4-sun: start with a K_4, add 'spikes' around the clique.



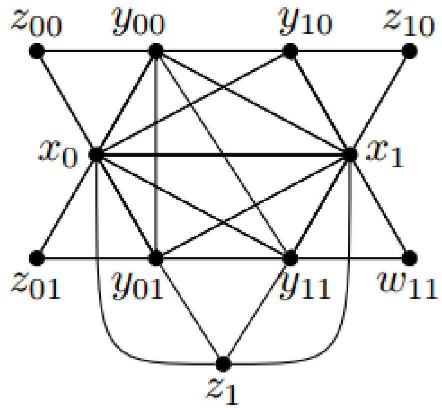
G_1



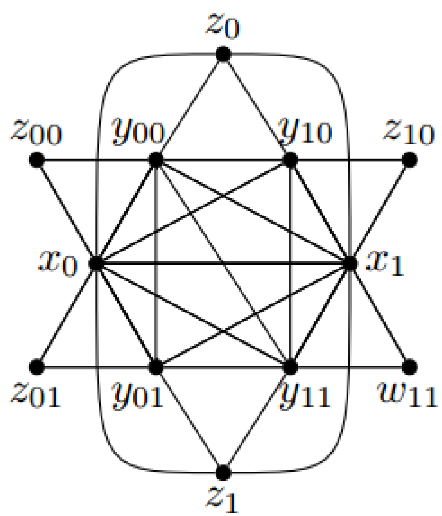
G_2



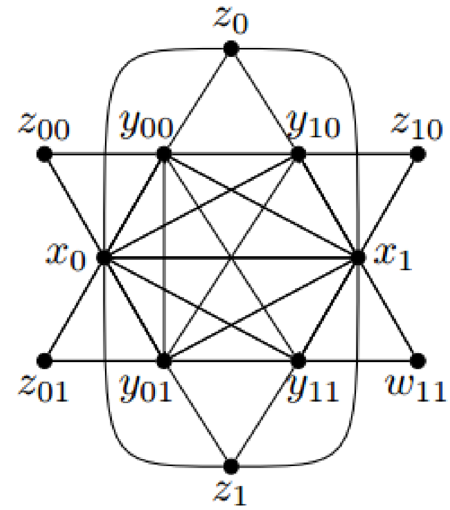
G_3



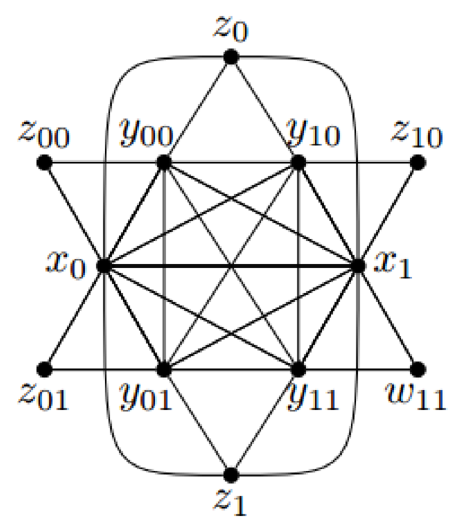
G_4



G_5



G_6



G_7

What is known?

Algorithmic perspective

Given a graph G, decide:

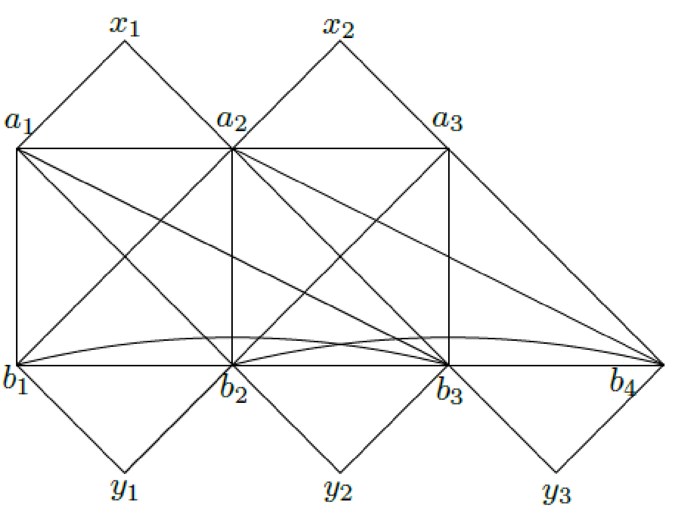
- whether G is a leaf power.
  - The complexity is open
- whether G is a k-leaf power, where k is given.
  - The complexity is open.
- whether G is a k-leaf power, where k is fixed.
  - In P for $k \leq 5$, open for $k \geq 6$.

In this work

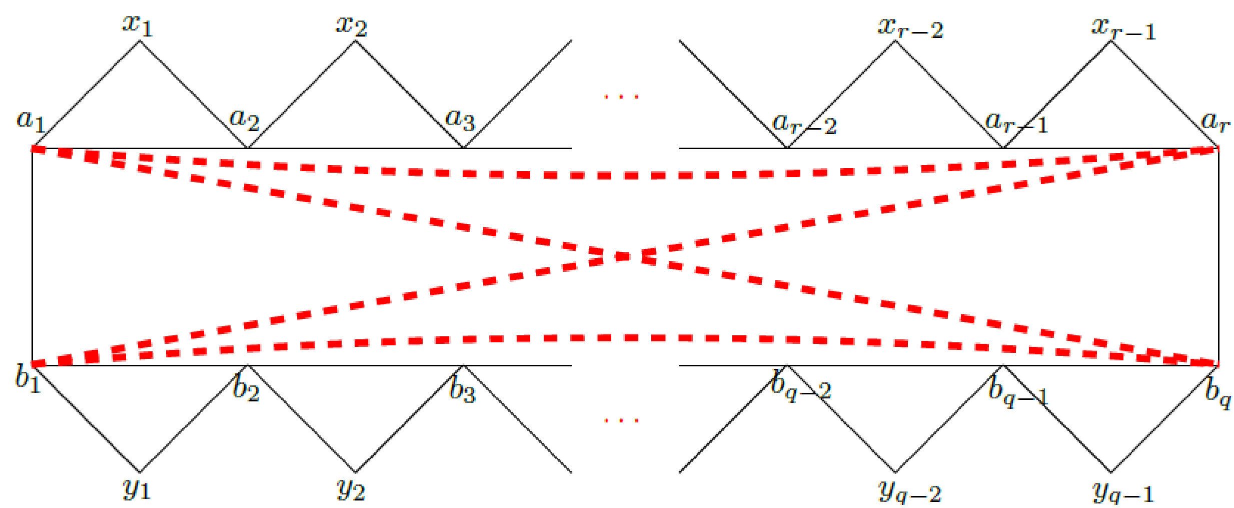
We show that leaf powers cannot be characterized by strongly chordality + a finite set of forbidden subgraphs.

- There exists an infinite family $G_{r,q}$ of (minimal) strongly chordal graphs that are not leaf powers.
- We establish a connection with leaf powers and **quartet compatibility**.

Deciding if a **chordal** graph G is **$G_{r,q}$-free** is NP-hard.



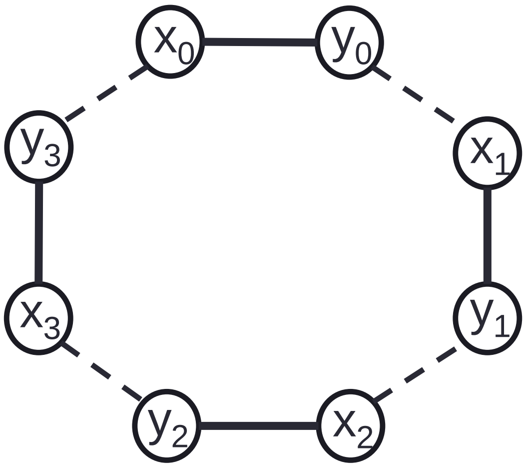
$G_{3,4}$



$G_{r,q}$

Alternating cycles

A sequence of vertices $x_0,y_0,x_1,y_1, \ldots,x_{c-1},y_{c-1}$ forms an alternating cycle if x_iy_i share an edge and y_ix_{i+1} do not (for all i, addition modulo c). *The other edges could be anything.*



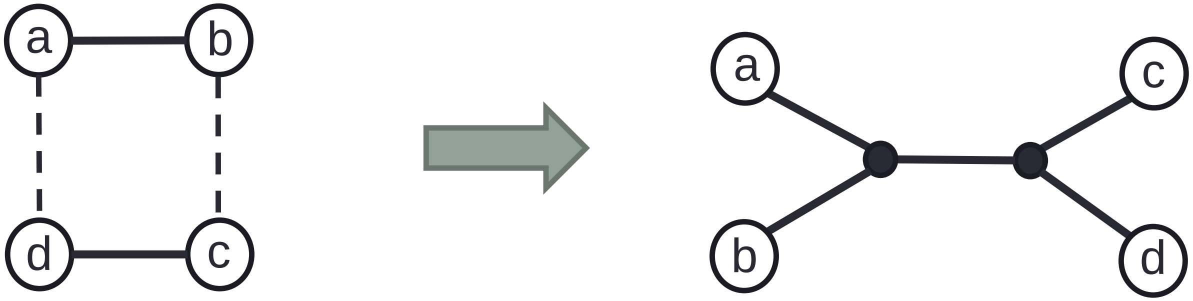
8-alternating cycle

- - - - Non-edges

4-alternating cycles and quartets

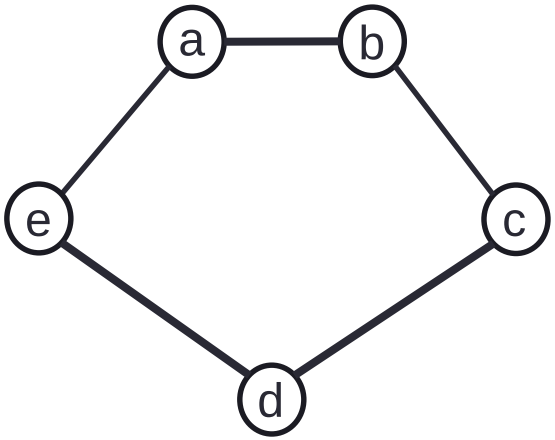
Lemma: if G is a leaf power and G contains the 4-alternating cycle a,b,c,d, then **any leaf root for G** must contain the **ab|cd quartet**.

- Meaning that in T, the a-b path and the c-d path share no vertex.



Why leaf powers are chordal?

A graph is chordal if all of its cycles on ≥ 4 vertices have a chord.



If G has this as an induced subgraph,
we have the alternating cycles
a,b,d,c => ab|cd quartet
b,c,e,d => bc|de quartet
c,d,a,e => ae|cd quartet
d,e,b,a => ab|de quartet

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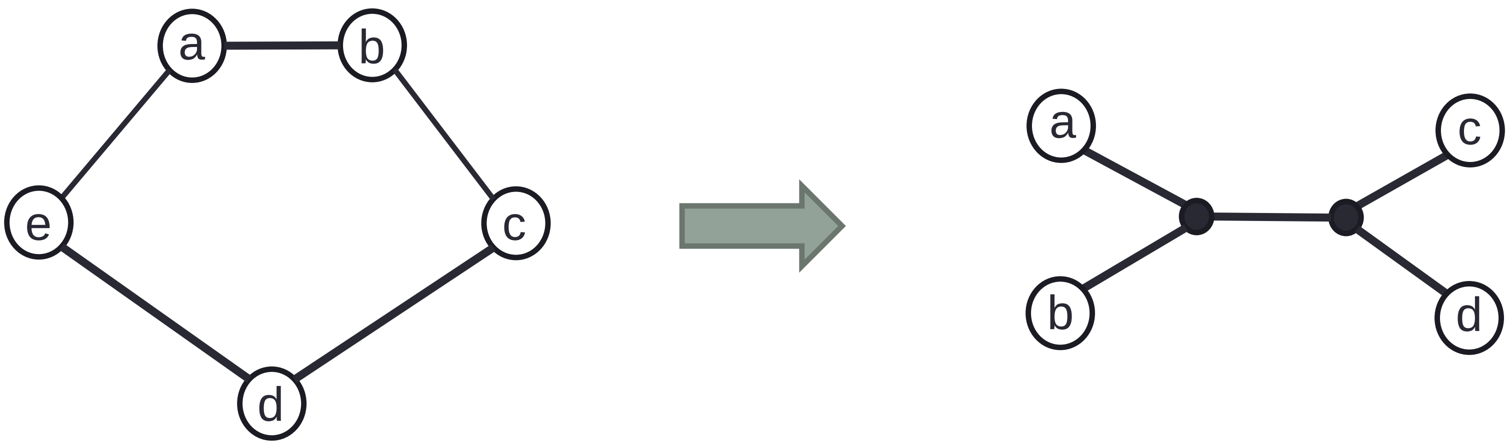

If G has this as an induced subgraph,

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a,b,d,c => **ab|cd** quartet

b,c,e,d => bc|de quartet

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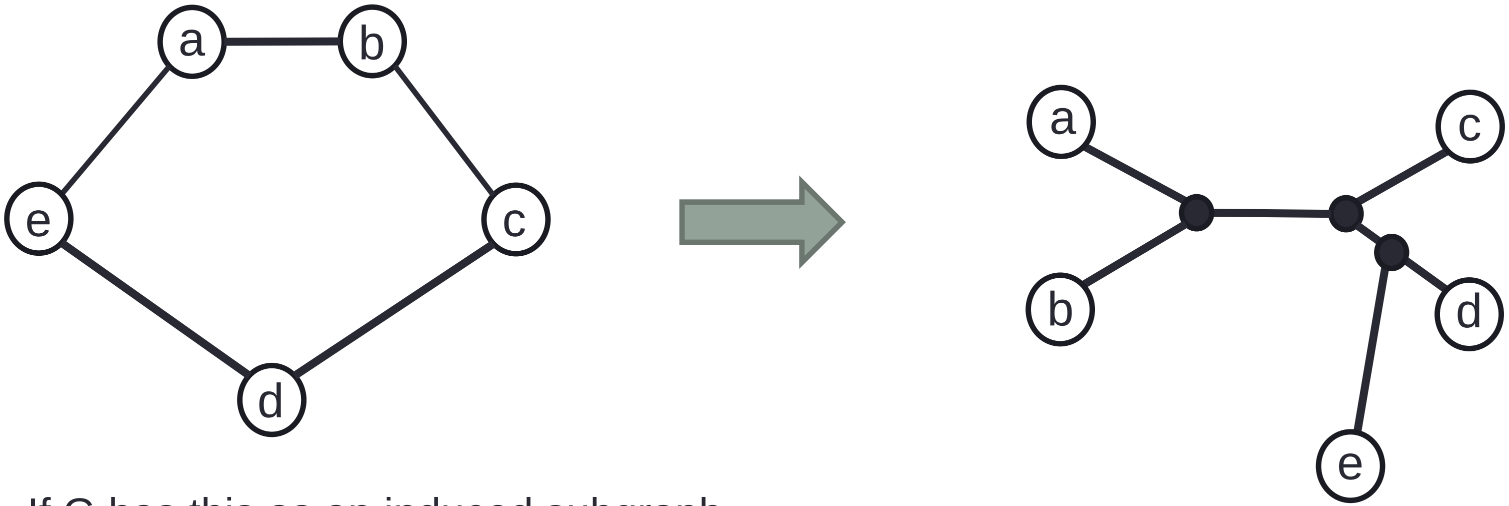

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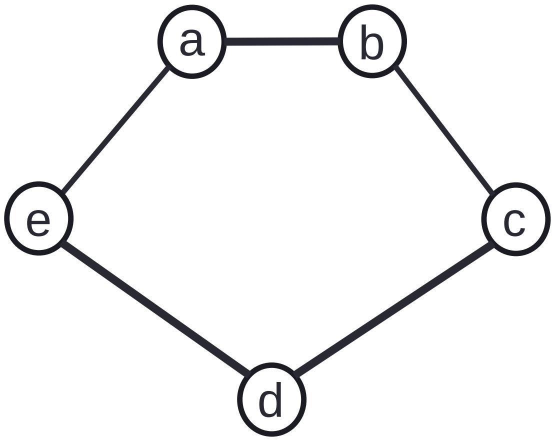
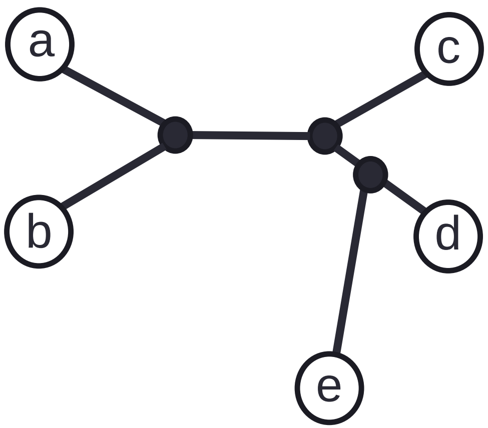


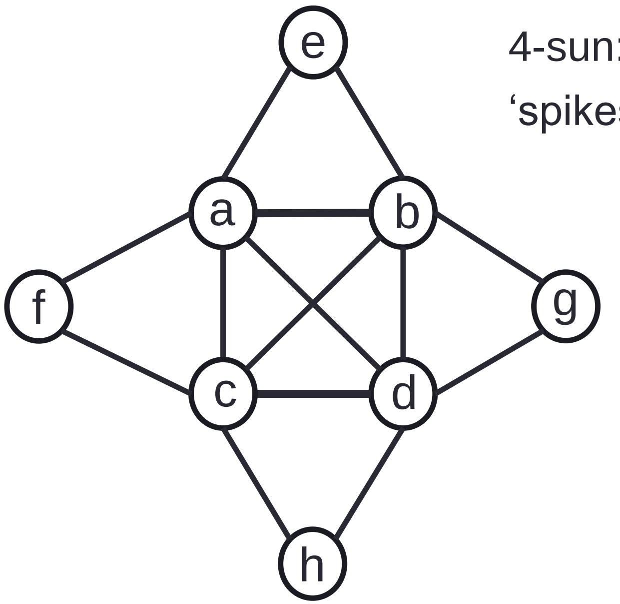


If G has this as an induced subgraph,
we have the alternating cycles
a,b,d,c => ab|cd quartet
b,c,e,d => bc|de quartet
c,d,a,e => **ae|cd** quartet … STUCK
d,e,b,a => ab|de quartet

No tree can satisfy the 4-alternating cycles of non-chordal graphs =>
cycles are forbidden induced subgraphs.

Why leaf powers are strongly chordal?

A graph is strongly chordal if it is chordal and sun-free.



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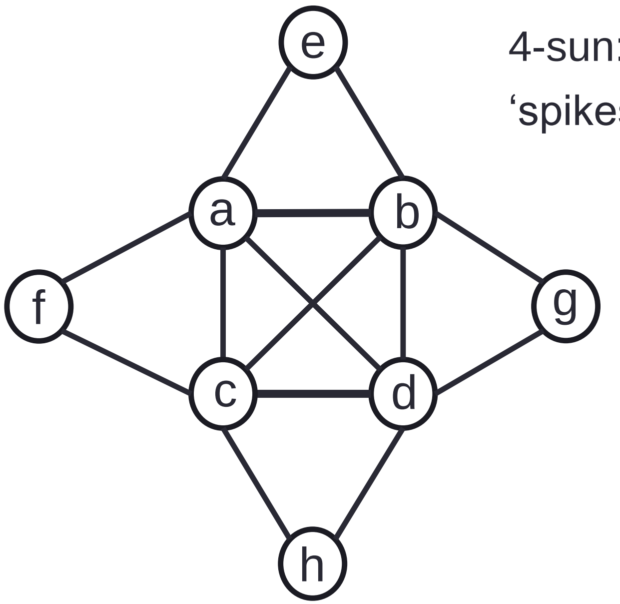

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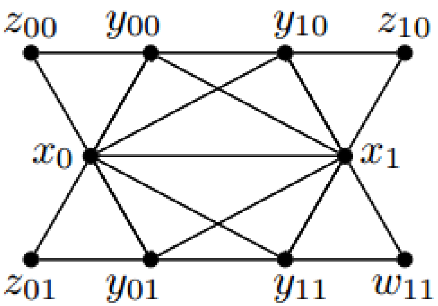
ae|ch + ae|dh + be|ch + be|dh => ab|cd

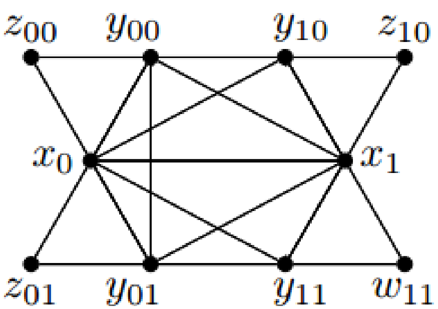
af|bg + af|dg + cf|bg + cf|dg => ac|bd

No tree can contain both these quartets => 4-suns are forbidden induced subgraphs of leaf powers.

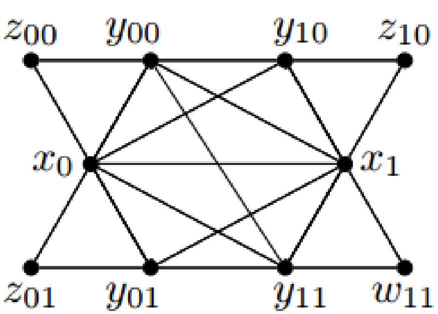
Same argument works for k-suns, k ≥ 4. Does not work for k = 3 (need to consider 6-alternating cycles).



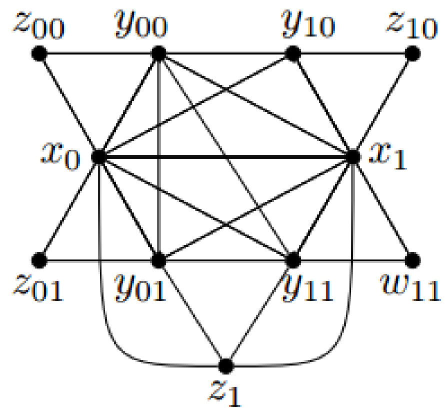
G_1



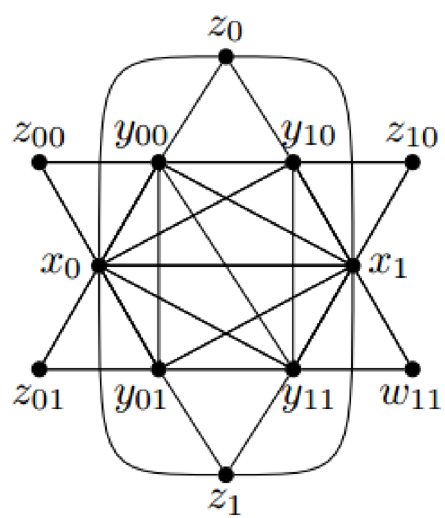
G_2



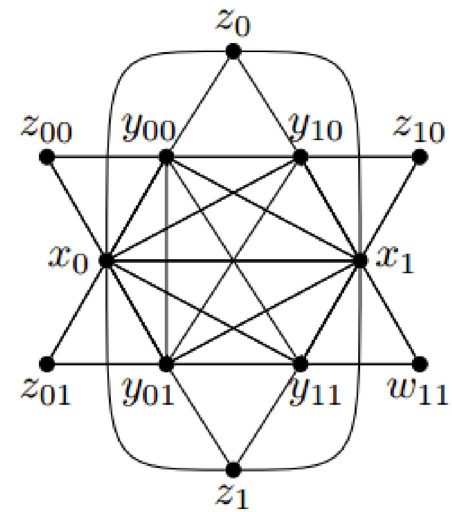
G_3



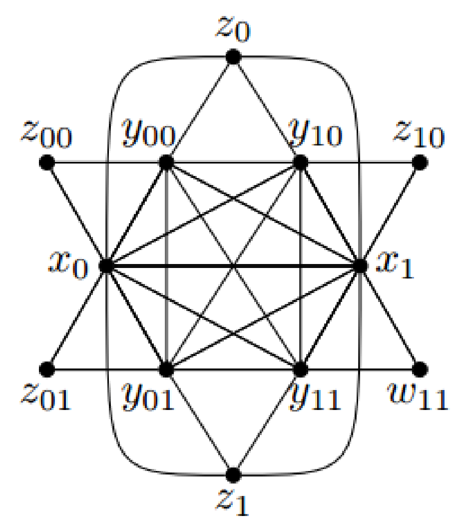
G_4



G_5



G_6



G_7

New examples of non-leaf powers

Theorem (Shutters, Vakati, Fernandez-Baca, 2012)
For any integer r, q ≥ 3, the set of quartets
$Q = \{a_i a_{i+1} \mid b_j b_{j+1} : 1 \leq i \leq r, 1 \leq j \leq q\} \cup \{a_1 b_1 \mid a_r b_q\}$ is incompatible.
Moreover, removing any quartet from Q makes it compatible.

Goal: for each r, q ≥ 3, construct a strongly chordal graph $G_{r,q}$ whose required set of quartets is Q, such that $G_{r,q} – v$ is a leaf power, for any v.
=> Provides an infinite family of **minimal** non-leaf powers that are strongly chordal.

New examples of non-leaf powers

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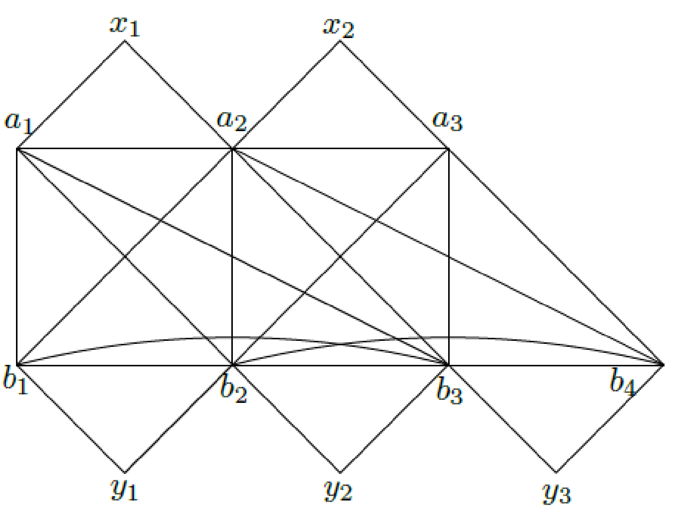
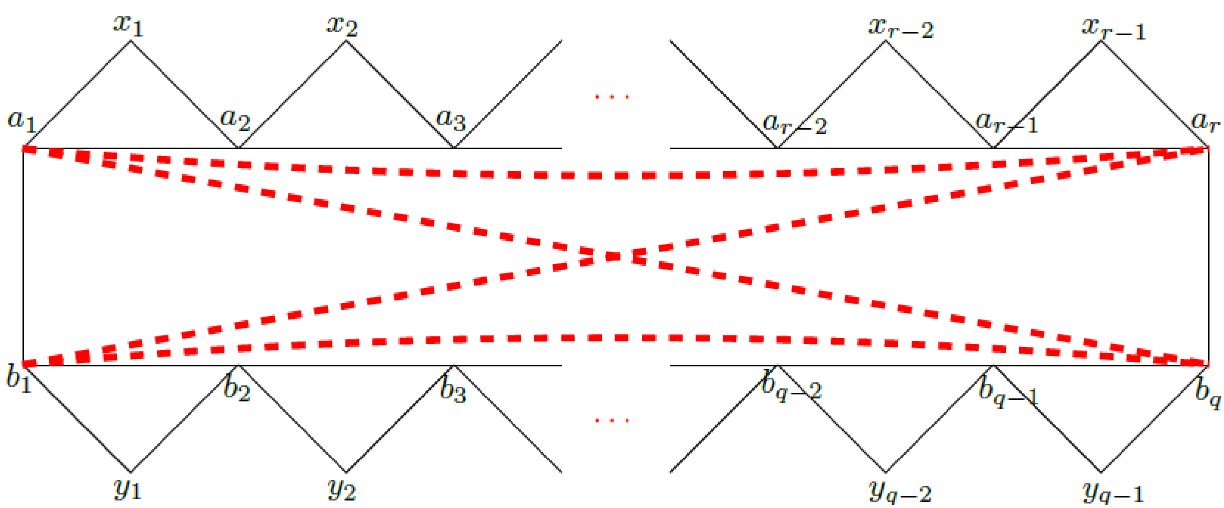


$G_{3,4}$



$G_{r,q}$

$a_i x_i \mid b_j y_j + a_i x_i \mid b_{j+1} y_j + a_{i+1} x_i b_j y_j + a_{i+1} x_i b_{j+1} \Rightarrow a_i a_{i+1} \mid b_j b_{j+1}$
And the 4-alternating cycle $a_1, b_1, b_q, a_r \Rightarrow a_1 b_1 \mid a_r b_q$

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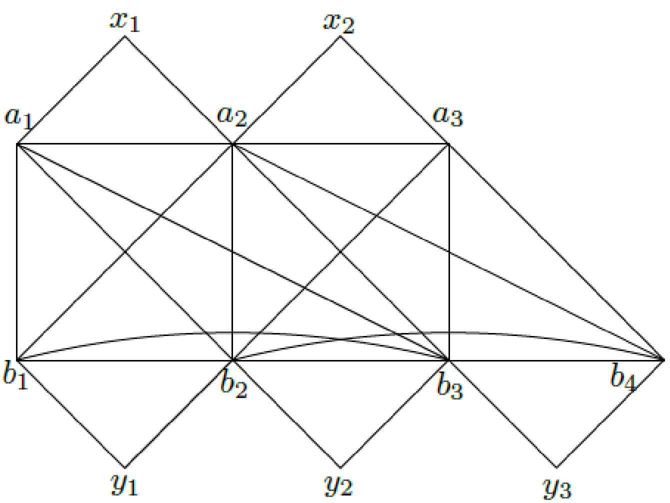
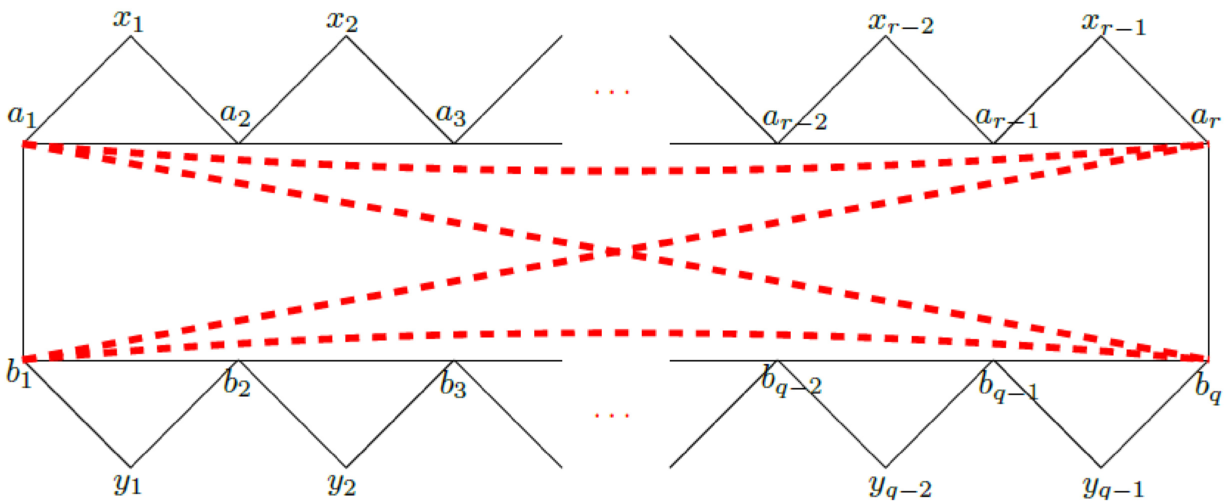


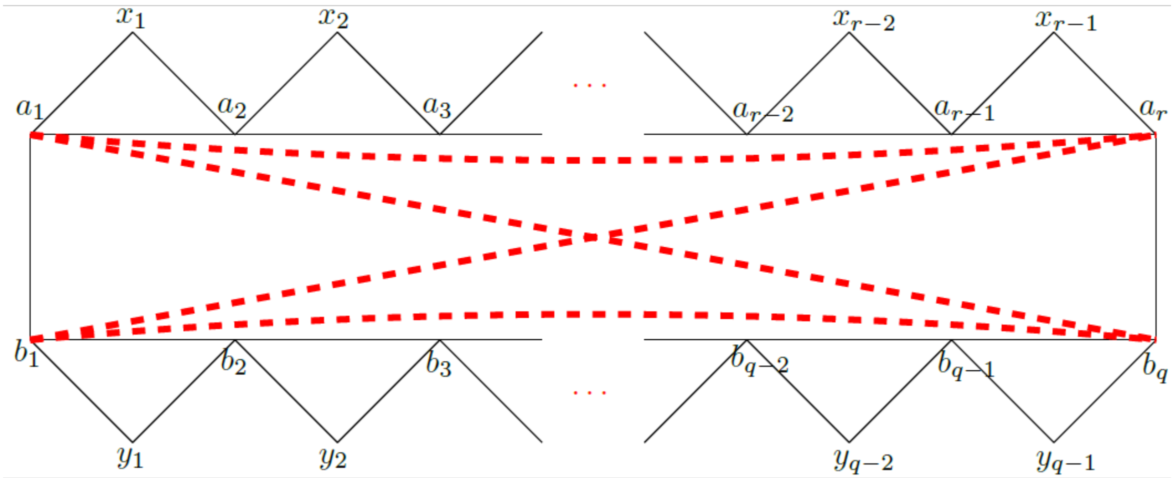
$G_{3,4}$

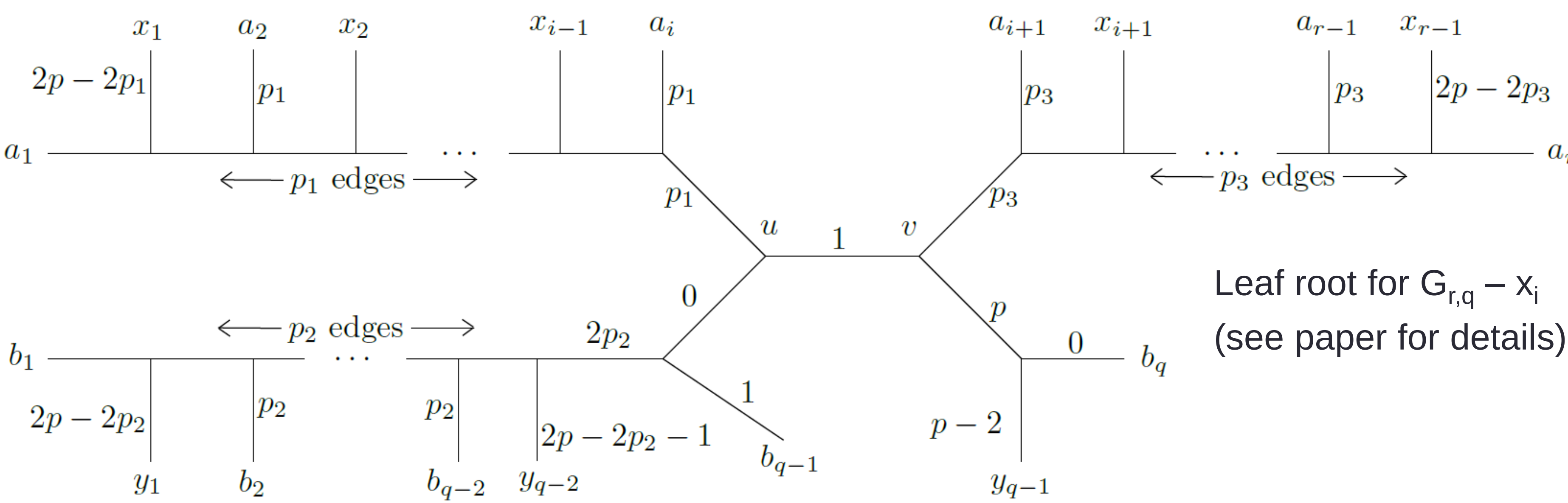


$G_{r,q}$

- Can be shown to be strongly chordal by a **simple elimination ordering.**
- **Minimality** requires constricting a leaf root for each $G_{r,q}$ - v

$G_{r,q}$ – v is a leaf power

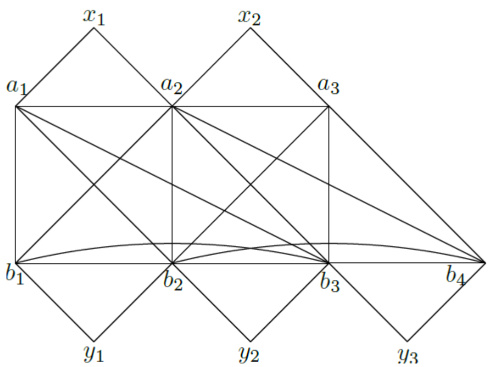




Leaf root for $G_{r,q} - x_i$
(see paper for details)

Detecting copies of $G_{r,q}$ in a graph

Given a graph G, can we detect whether it contains a copy of $G_{r,q}$?



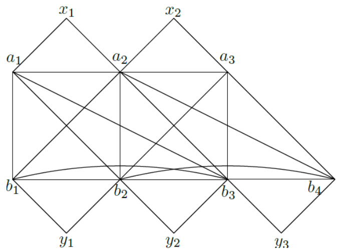
Detecting copies of $G_{r,q}$ in a graph

Given a graph G, can we detect whether it contains a copy of $G_{r,q}$?

Theorem: deciding if G contains an induced $G_{r,q}$ for some $r,q \geq 3$ is NP-hard, even if G is a chordal graph.

(reduction from
Does there exist a chordless cycle between two specified vertices s, t in a bipartite graph such that s, t have degree two and share no neighbor and are in the same part of the bipartition).

$G_{r,q}$-freeness is the first known property of leaf powers that we do not (yet) know how to check in polynomial time.
Could $G_{r,q}$ be, conceivably, used to show the hardness of recognizing leaf powers?



Conclusion

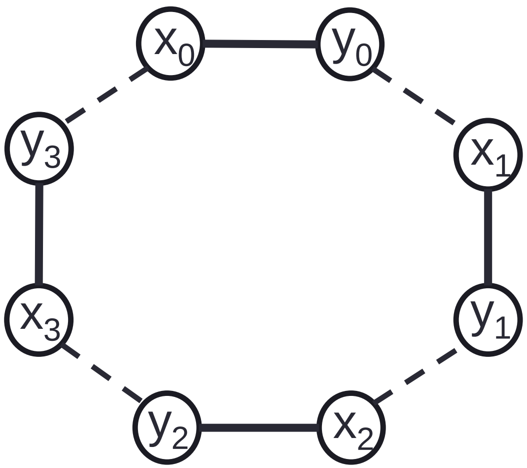
- Can alternating cycles provide more insight on the class of leaf powers?
- **Conjecture**: a graph G is a leaf-power iff there is a tree T that can satisfy each of its (4,6)-alternating cycles.
- Are there other strongly chordal non-leaf powers? (short answer: yes)
  - Can we characterize them?
- Can we find copies of $G_{r,q}$ in strongly chordal graphs?

- Still open: recognize k-leaf powers for fixed k
  - k-leaf powers have bounded clique-width
  - Unlike leaf powers, k-leaf powers may allow a characterization by strong chordality + a finite set of forbidden subgraphs.

Alternating cycles

A tree T **can satisfy** an alternating cycle C = $(x_0, y_0, x_1, y_1, \ldots, x_{c-1}, y_{c-1})$ if the edges of T can be weighted so that **there is a k such that $d_T(x_i, y_i) \leq k$ and $d_T(y_i, x_{i+1}) > k$** (for all i)

- In words, T can be a leaf power if we only care about the edges/non-edges of C.



8-alternating cycle

- - - - Non-edges

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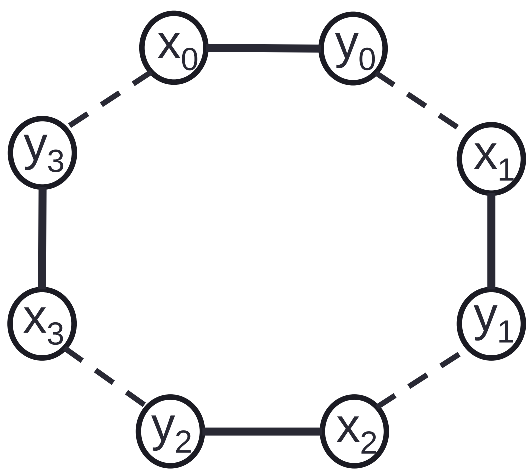

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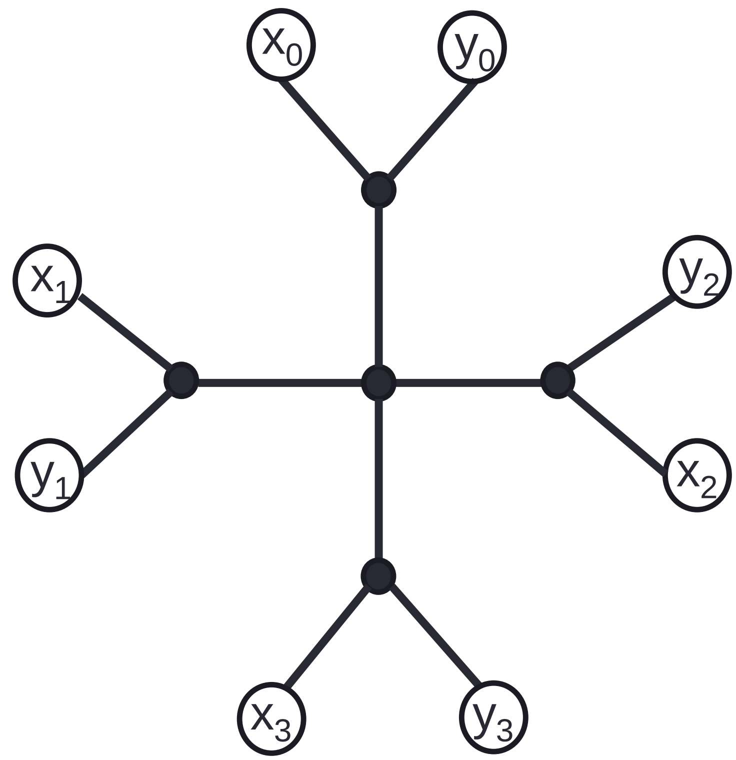

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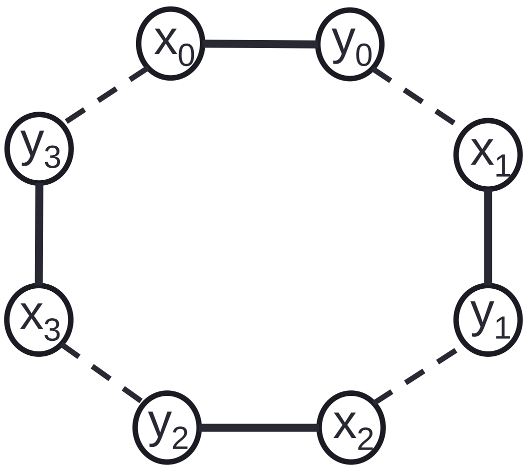

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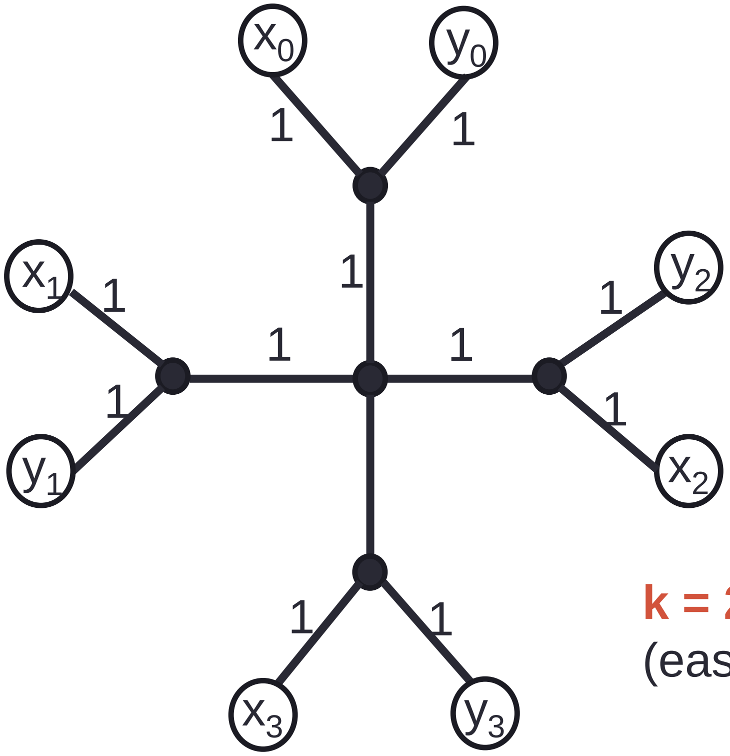
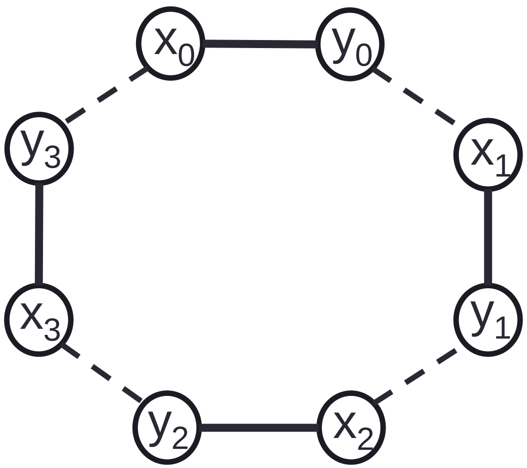


k = 2

(easy right?)

Alternating cycles



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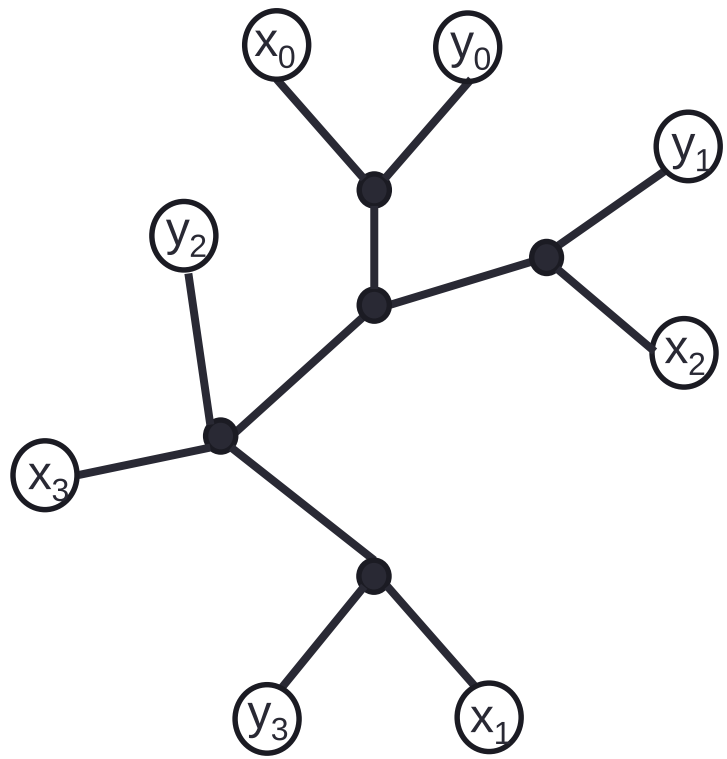

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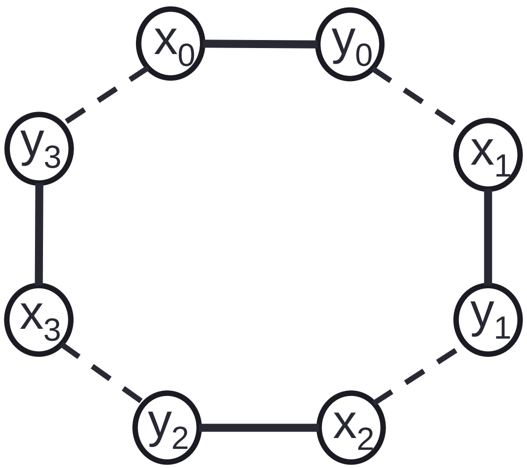

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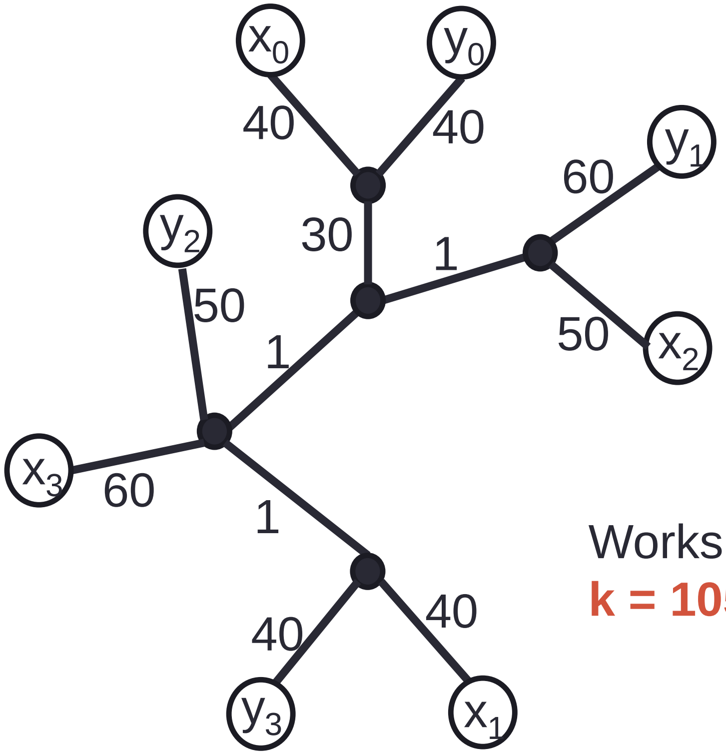

Works with
k = 105

What's the point?

Proposition: if G is a leaf power, then there exists a tree T that can satisfy every alternating cycle of G.

(*Proof: if G is a leaf power, any leaf root T for G must, in particular, satisfy the edges/non-edges of the alternating cycles of G.*)

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As it turns out, **every** graph that is known to not be a leaf power fails to meet this requirement.

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- **Conjecture:** a graph G is a leaf power **if and only if** there exists a tree T that can satisfy every alternating cycle of G.

In fact, every known non-leaf power has no tree that can satisfy all of its **4-alternating cycles,** with one single exception: the 3-sun, for which no tree can satisfy its **4 and 6-alternating cycles**.

The above proposition can also be used to build **new examples** of non-leaf powers.

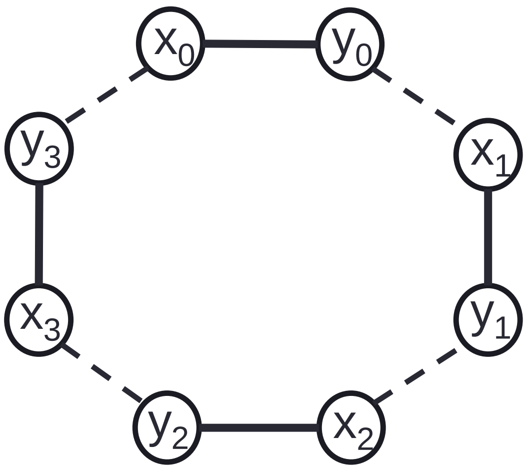
Alternating cycles

For each edge xy of C (e.g. x_0y_0), mark the edges of T on the x-y path by '+'

For each non-edge xy of C (e.g. y_0x_1), mark the edges of T on the x-y path by '-'

Lemma

The tree T can satisfy C iff it has some edge marked by strictly more '-' than '+'



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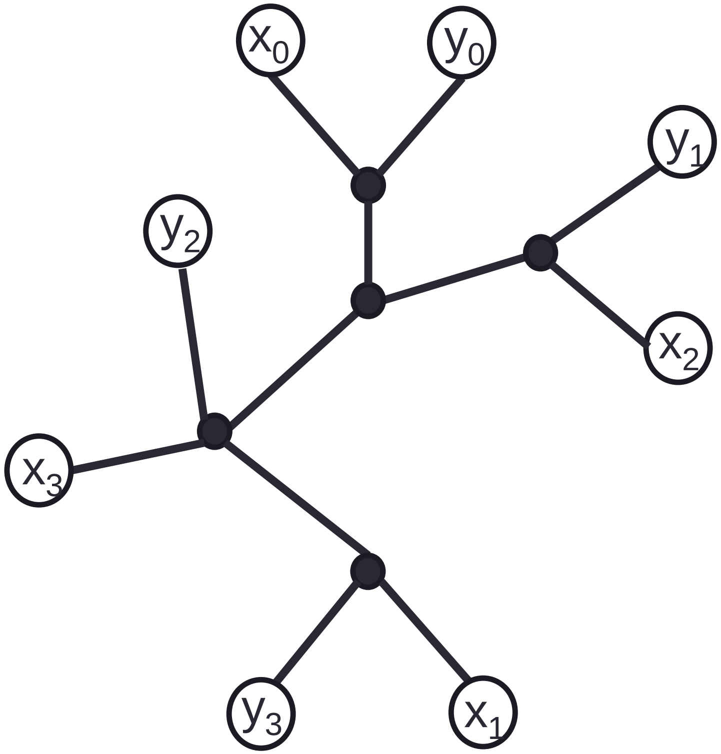

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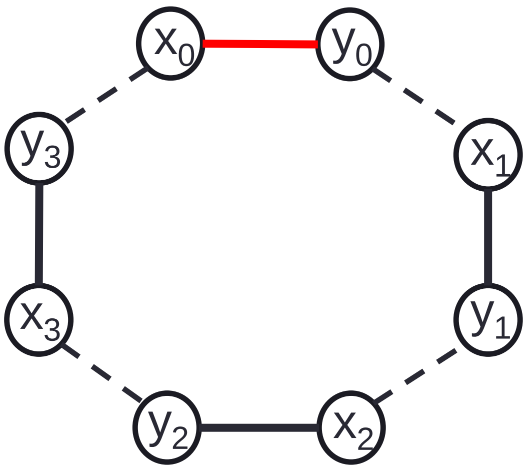

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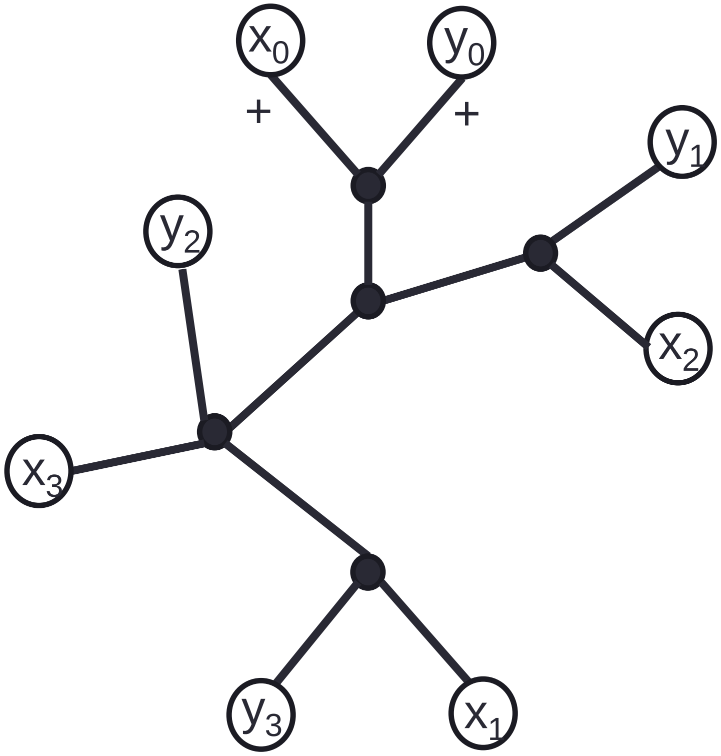

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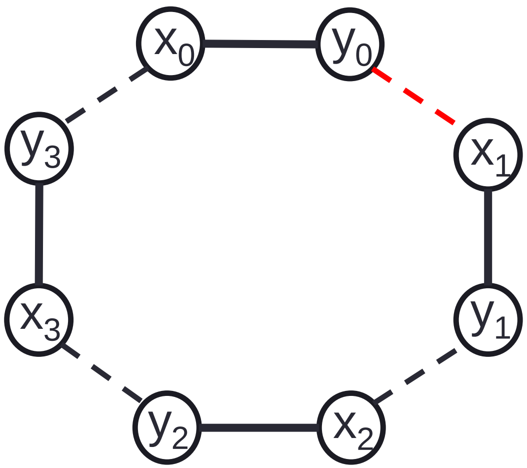

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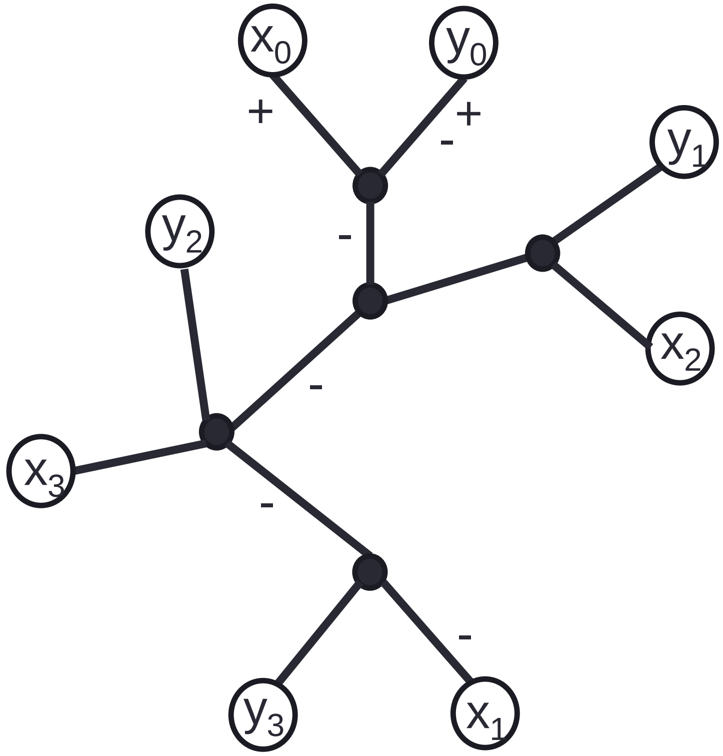

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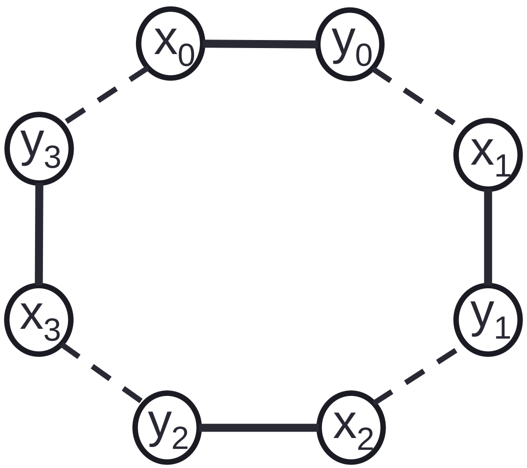

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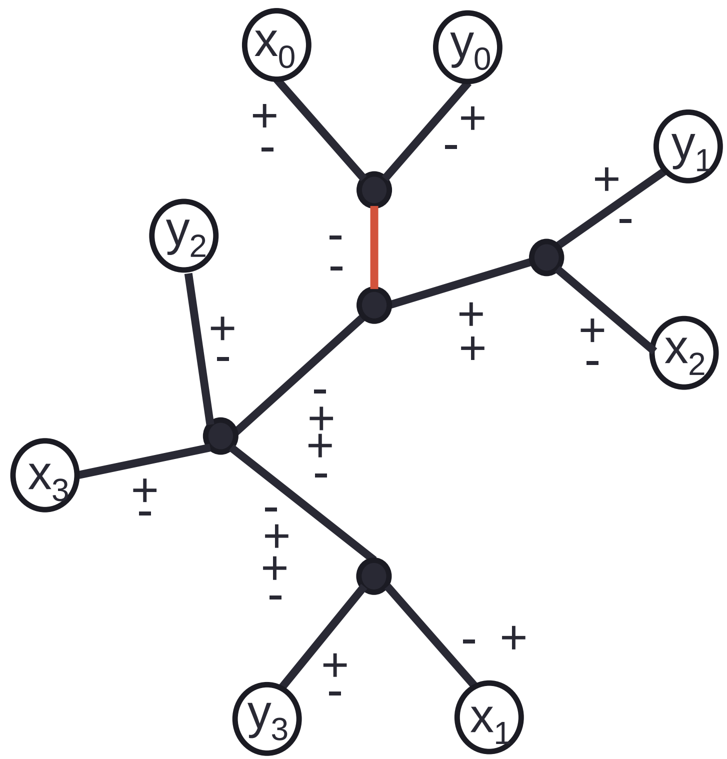

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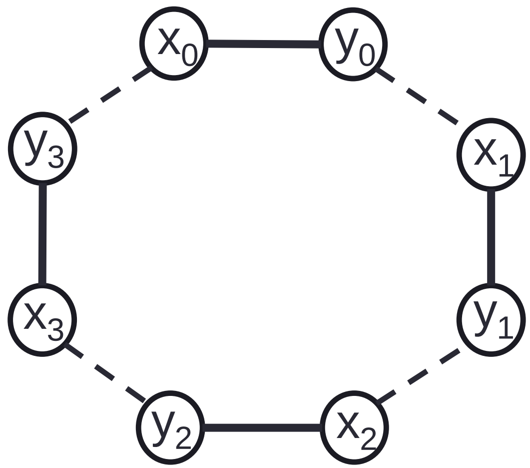

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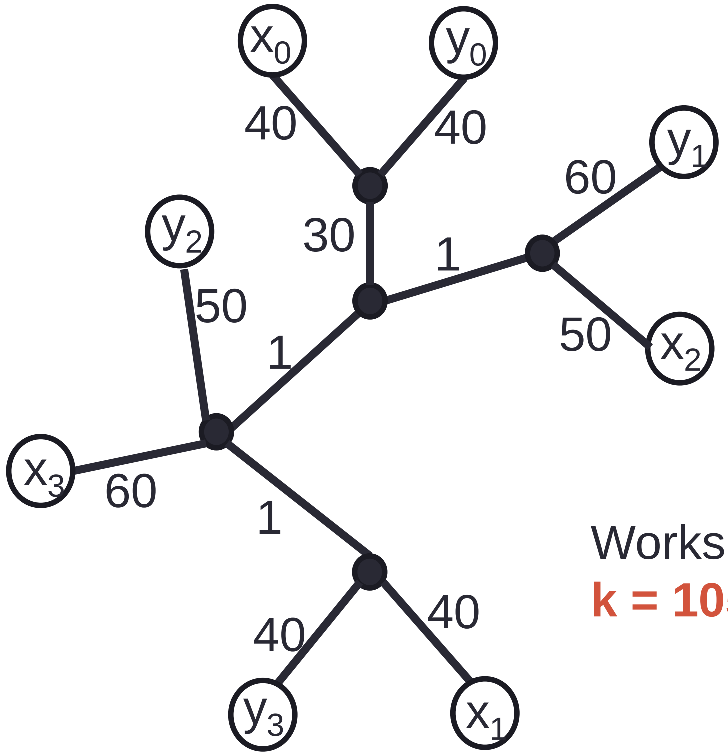
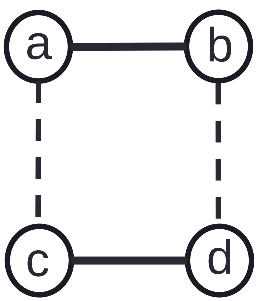


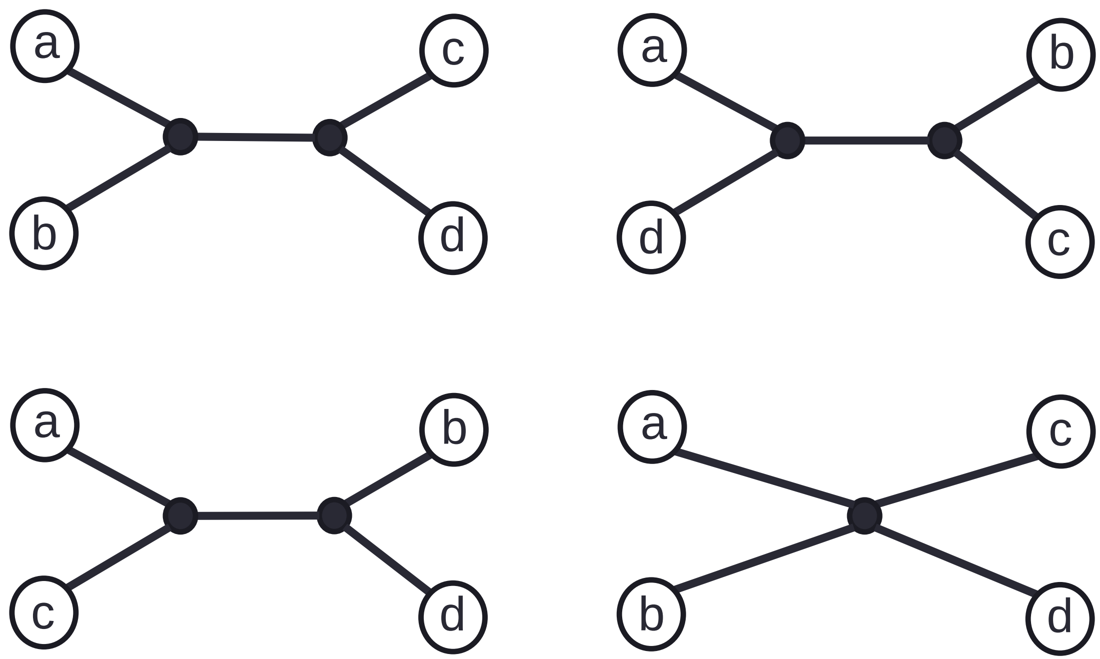
Works with

k = 105

4-alternating cycles and quartets

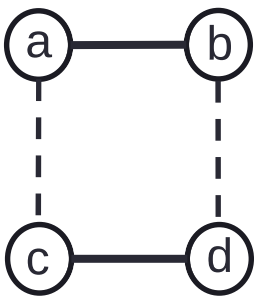


4-alternating cycle



Trees on 4 leaves

4-alternating cycles and quartets



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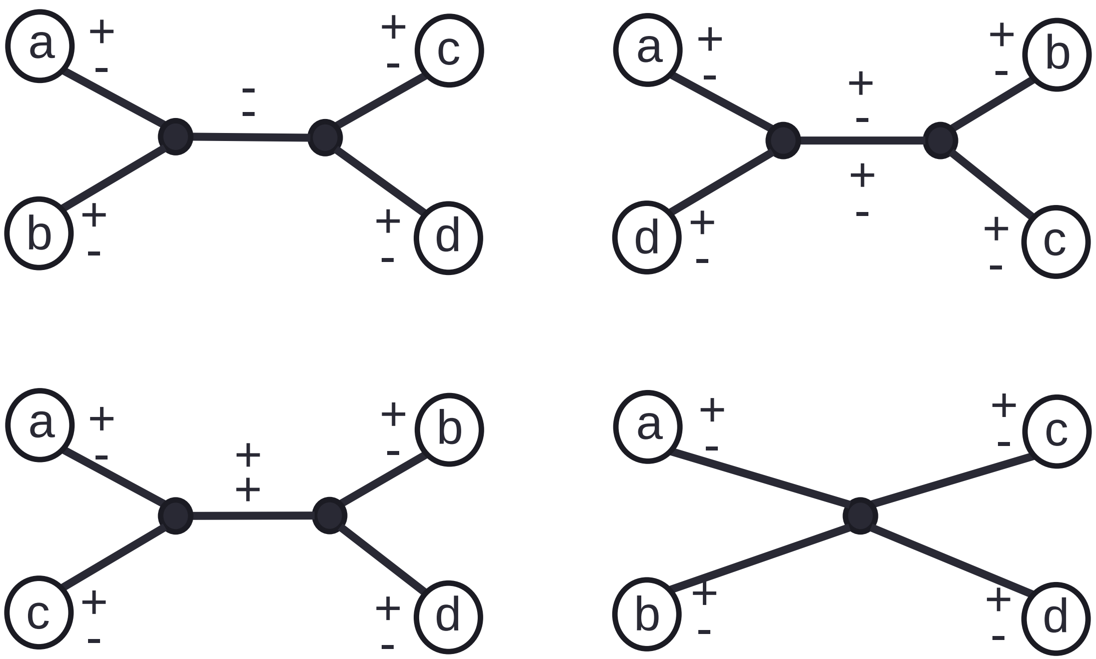

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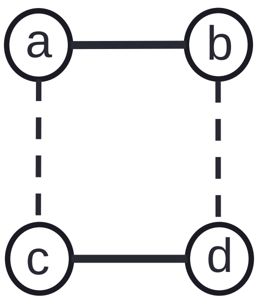

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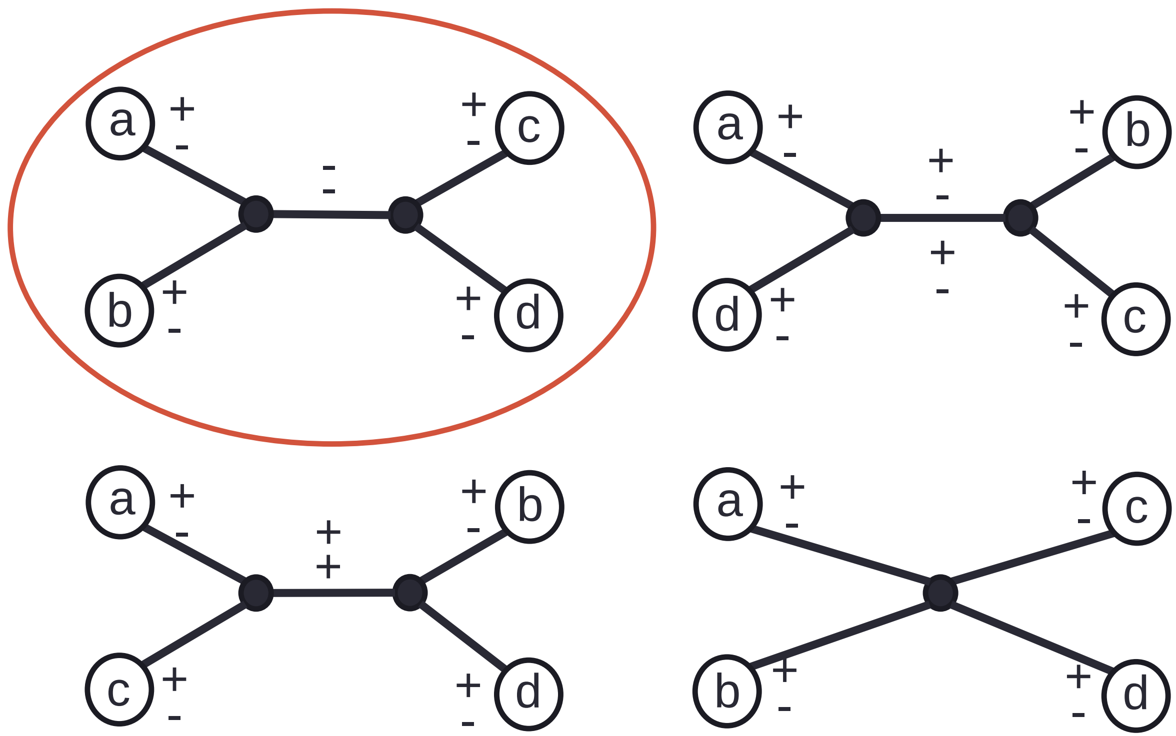

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