WILL WE EVER FIND A FORBIDDEN SUBGRAPH CHARACTERIZATION OF LEAF POWERS?

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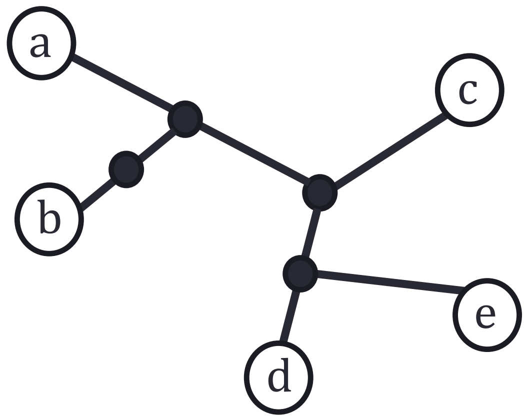
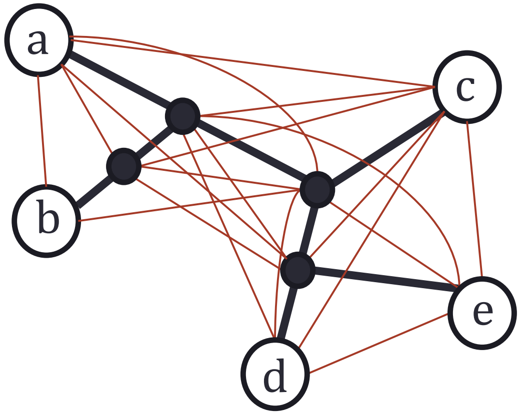
Tree T, leafset denoted $L(T)$, leaves have distinct labels

T :

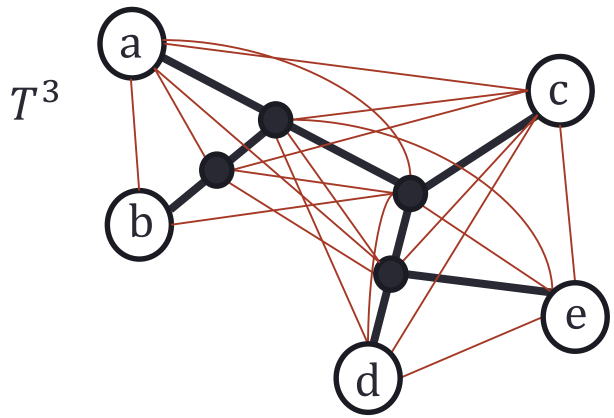


- T^k : k-th power of T
  - $V(T^k) = V(T)$
  - $uv \in E(T^k) \Leftrightarrow dist_T(u, v) \leq k$

T^3



- The ***k*-th leaf power** of T is $T^k[L(T)]$, the subgraph induced by the leaves.



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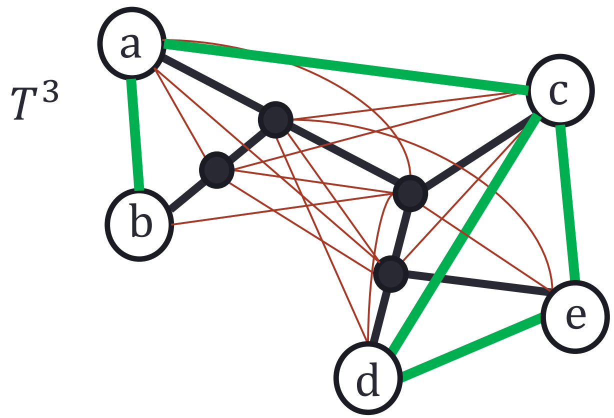

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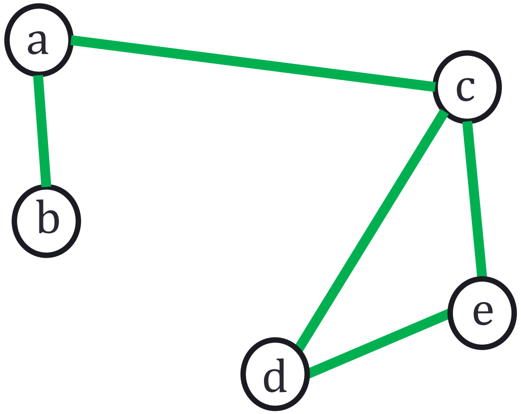
$T^3[L(T)]$



Definition
A graph G is a ***k*-leaf power** if there exists a tree T such that $G = T^k[L(T)]$

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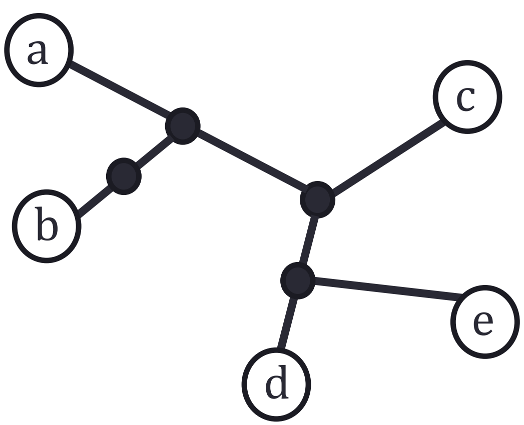

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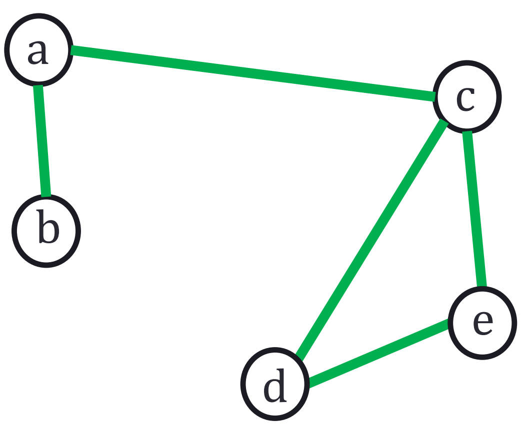

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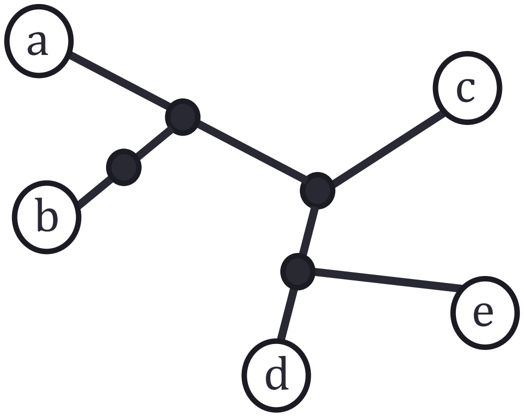

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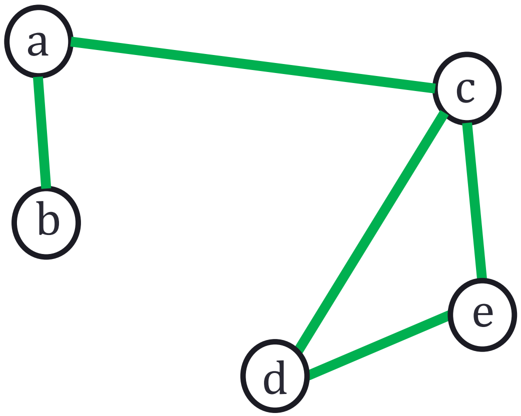

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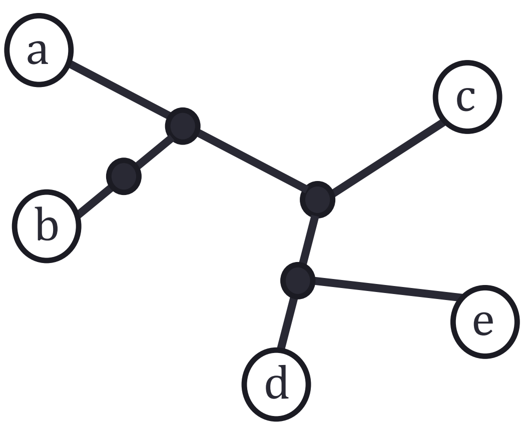

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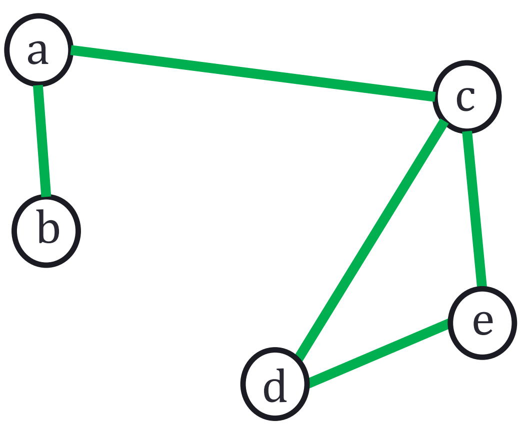

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A graph G is a **leaf power** if there exists k such that G is a k-leaf power .

Graph theoretical problems [Nishimura, Ragde, Thilikos, 2002]
- Provide a forbidden induced subgraph characterization of leaf powers.
- Provide a forbidden induced subgraph characterization of k-leaf powers, for every k.

Algorithmic problems
- Is deciding whether a graph is a leaf power in P?
- For fixed k, is deciding whether a graph is a k-leaf power in P?

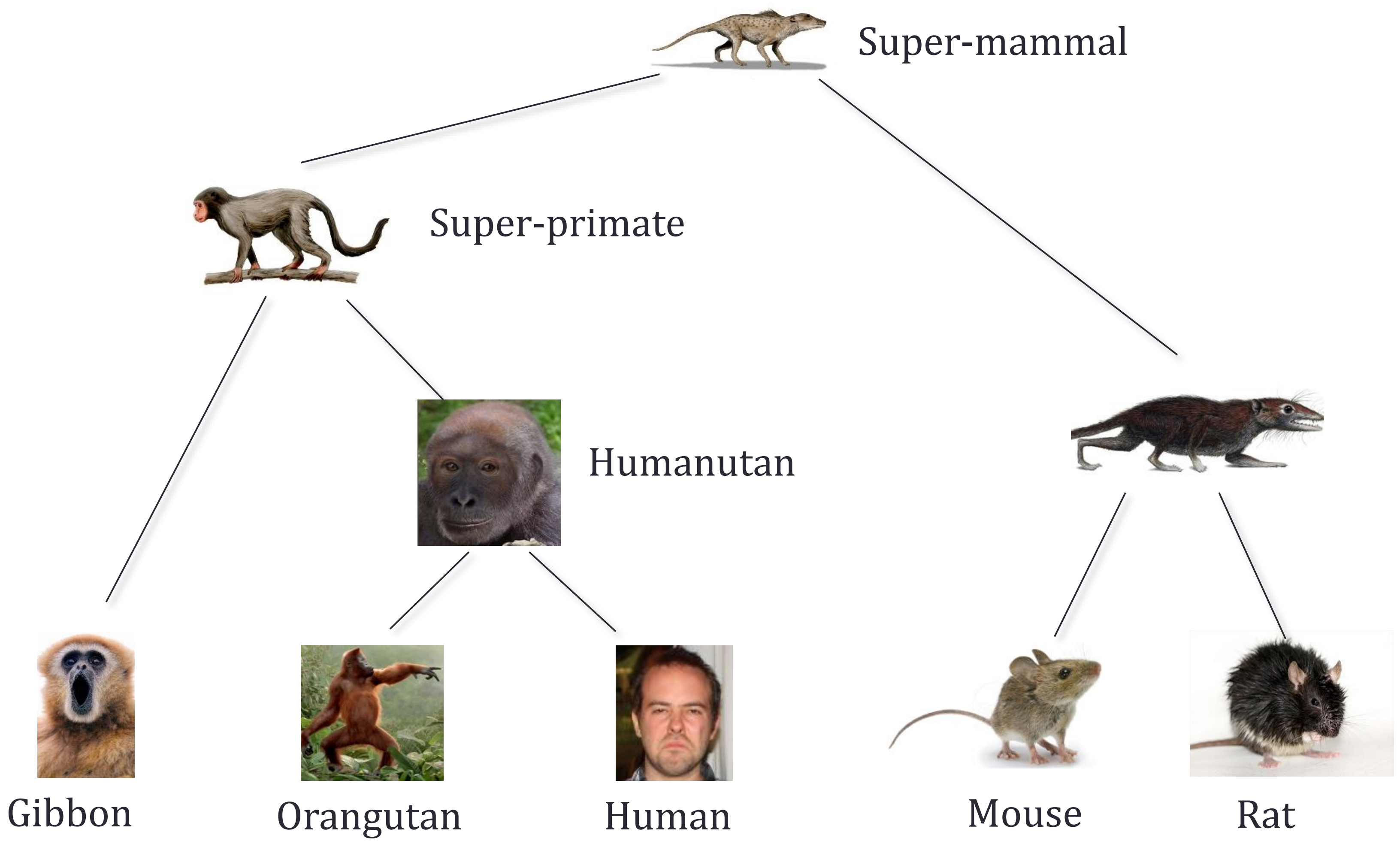
In this talk

Graph theoretical problems
- Provide a forbidden induced subgraph characterization of leaf powers.

Conjecture (Nevries & Rosenke, 2015)
There are only a finite number of minimal strongly chordal graphs that are not leaf powers.
(implication: easy poly-time algorithm)

(L, 2018)
There is an infinity of minimal strongly chrodal graphs that are not leaf powers.
(implication: still no easy poly-time algorithm)

Motivation


Super-mammal
Super-primate
Humanutan
Gibbon
Orangutan
Human
Mouse
Rat

Motivation

Gibbon

Orangutan

Human

Mouse

Rat

Motivation

- Distance matrix.
- Noisy.
- Basic info: "near" or "far"
  - w.r.t some threshold

	G	O	H	M	R
G	0	0.33	0.25	0.52	0.61
O		0	0.31	0.61	0.36
H			0	0.55	0.56
M				0	0.2
R					0


Gibbon


Orangutan


Human


Mouse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threshold = 0.5


Gibbon


Orangutan


Human


Mouse


Rat

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Gibbon

Orangutan

Human

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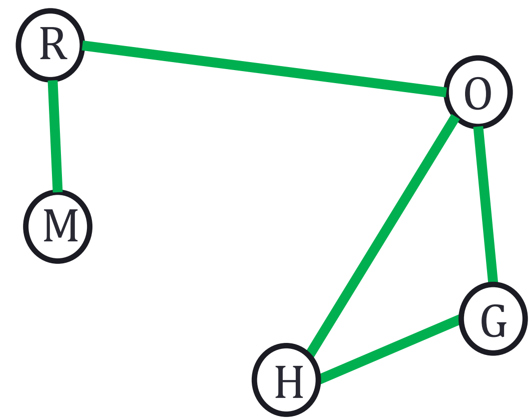

Gibbon

Orangutan

Human

Mouse

Rat

Motivation

- This graph "makes sense" if we can find a tree T in which neighbors are "close", non-neighbors are "far".
- Does there exist k and a tree T with leafset $V(G)$ satisfying

$$uv \in E(G) \Leftrightarrow dist_T(u,v) \leq k$$

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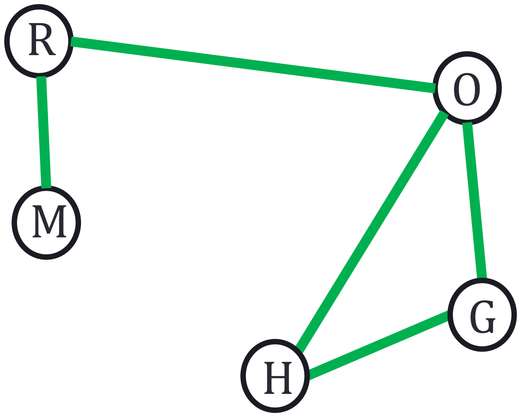

Motivation

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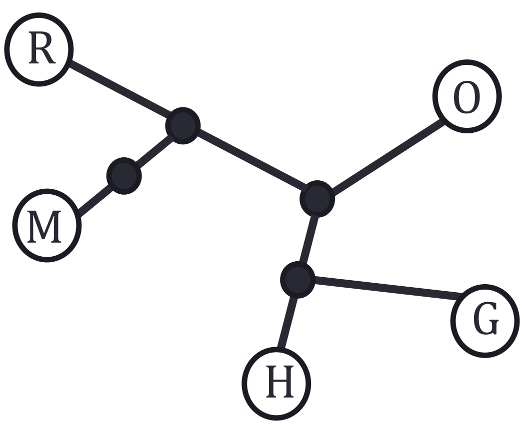
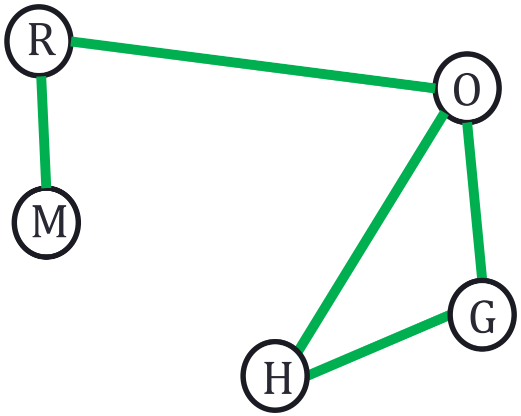




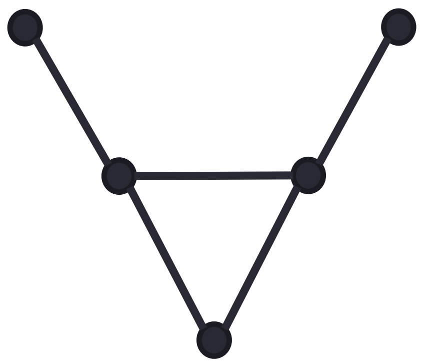
So what do we know about leaf powers?

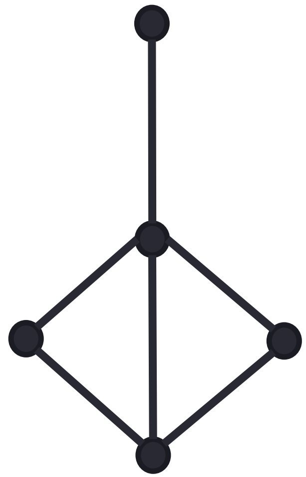
Proposition

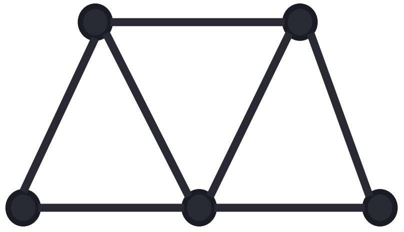
A graph G is a 2-leaf power if and only if it contains no induced P_3 (a path of length 2).

Theorem [Dom, 2006][Rautenbach, 2006]

A graph G is a 3-leaf power if and only if it is **chordal** and does not contain a bull, dart of gem as an induced subgraph.

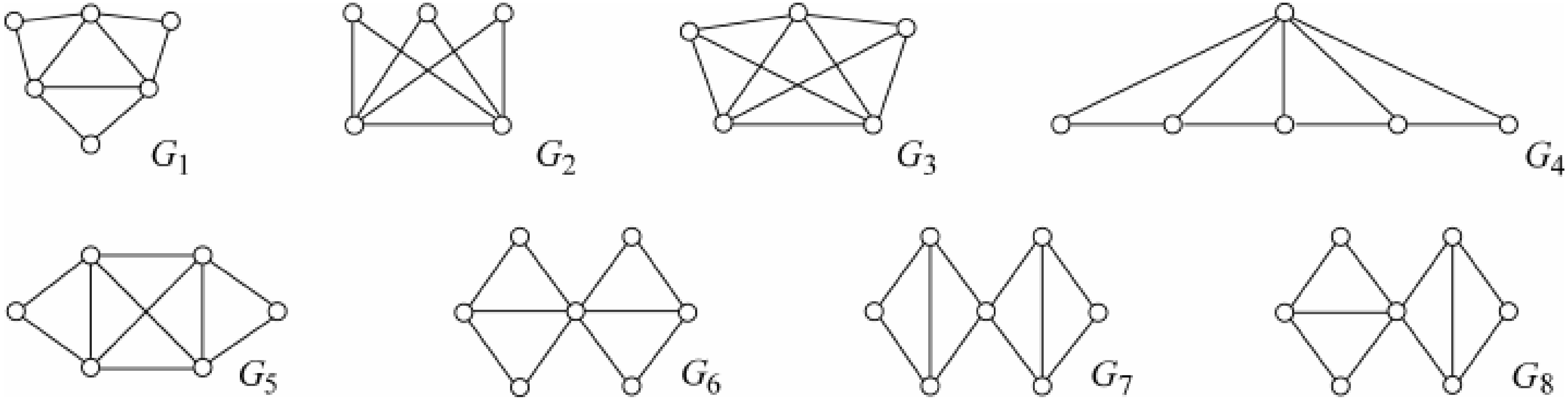






Theorem [Brandstädt and Hundt, 2008]

A graph G is a (twin-free) 4-leaf power if and only if it is **chordal** and does not contain any of these graphs:



Algorithmic results

Theorem [Chang & al., 2007]

5-leaf powers can be recognized in polynomial time.

Theorem [Eppstein, Havvaei, 2018]

k-leaf powers with bounded degeneracy d can be recognized in time $f(k,d)poly(n)$.

Theorem [Ducoffe, 2018]

6-leaf powers can be recognized in polynomial time.

[65 pages]

Results on leaf powers

- If G is a leaf power, then G is chordal (i.e. every cycle of length at least 4 has a shortcut).

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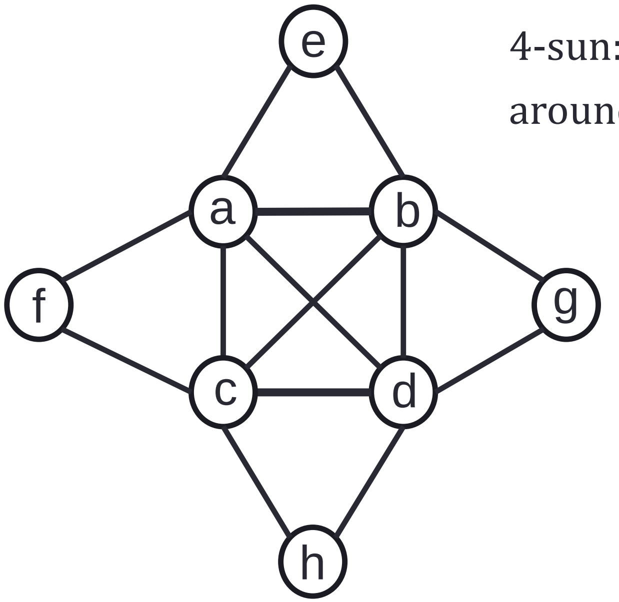
Results on leaf powers

- If G is a leaf power, then G is **strongly chordal**.

A graph is strongly chordal if it is chordal and k-sun-free, for every k ≥ 3.
A k-sun is the graph obtained by starting with a clique with vertices $v_1, \ldots vk$, and adding a common neighbor to each v_i, v_{i+1} pair.



4-sun: start with a K_4, add 'spikes' around the clique.

Open question (at the time)

Are leaf powers exactly the strongly chordal graphs?

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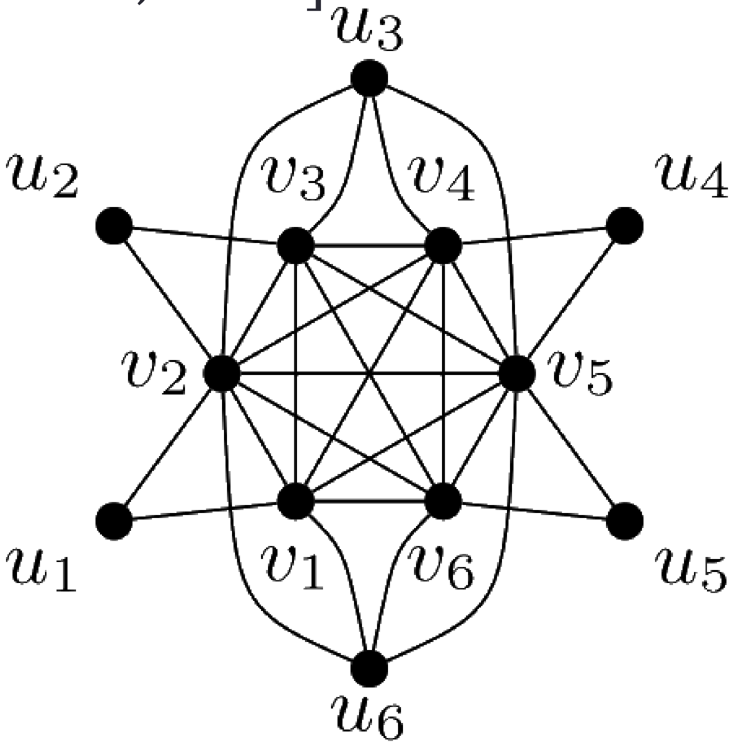
NO

[Brandstädt and Hundt., 2010]

Strongly chordal, not a leaf power



Open question

Is there only one (minimal) strongly chordal graph that is not a leaf power?

Minimal in the sense that removing a vertex makes the graph a leaf power.

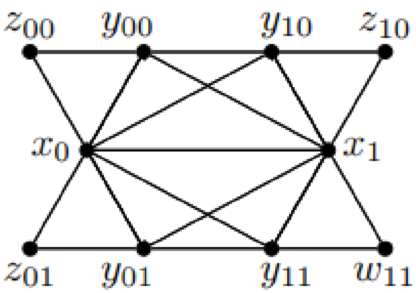
Open question

Is there only one (minimal) strongly chordal graph that is not a leaf power?

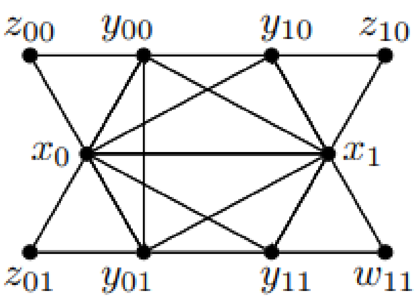
NO

There are at least 7 of them!

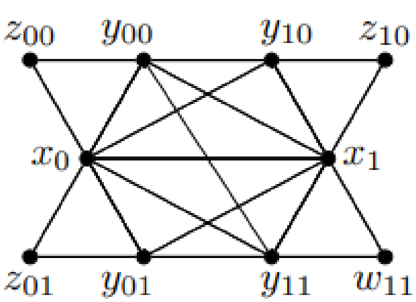
[Nevries & Rosenke, 2015]



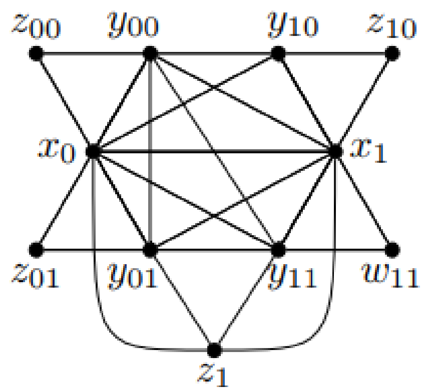
G_1



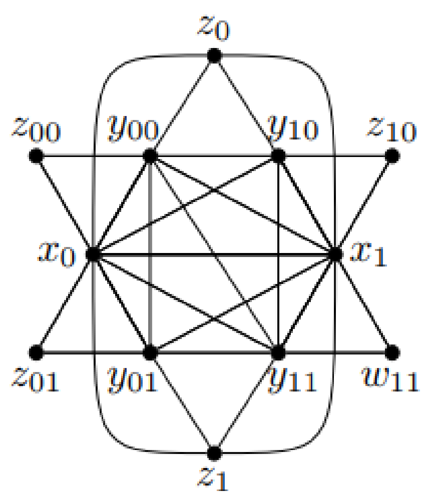
G_2



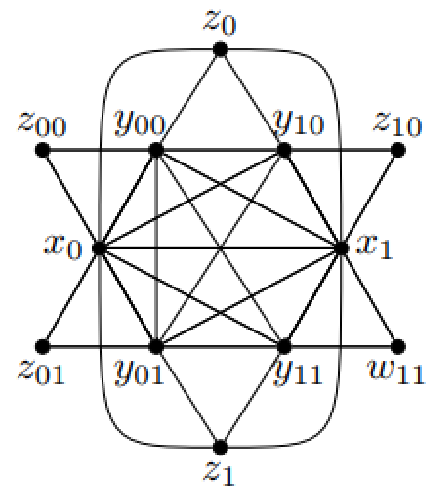
G_3



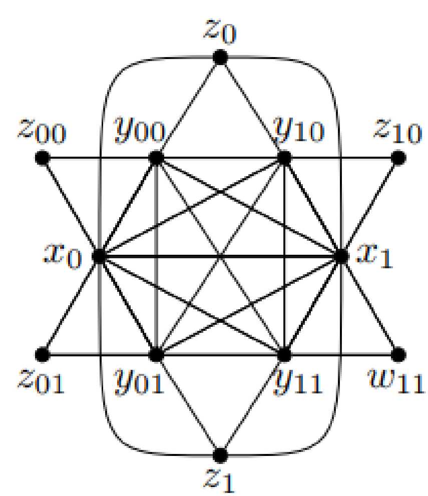
G_4



G_5



G_6



G_7

Conjecture

There are only a finite number of (minimal) strongly chordal graphs that are not leaf powers.

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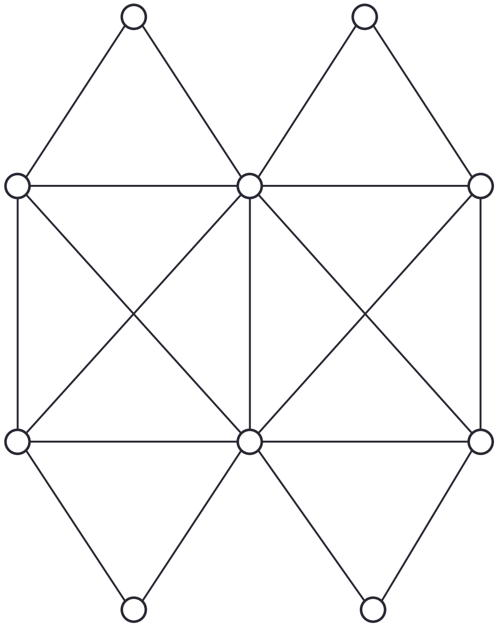
NO!

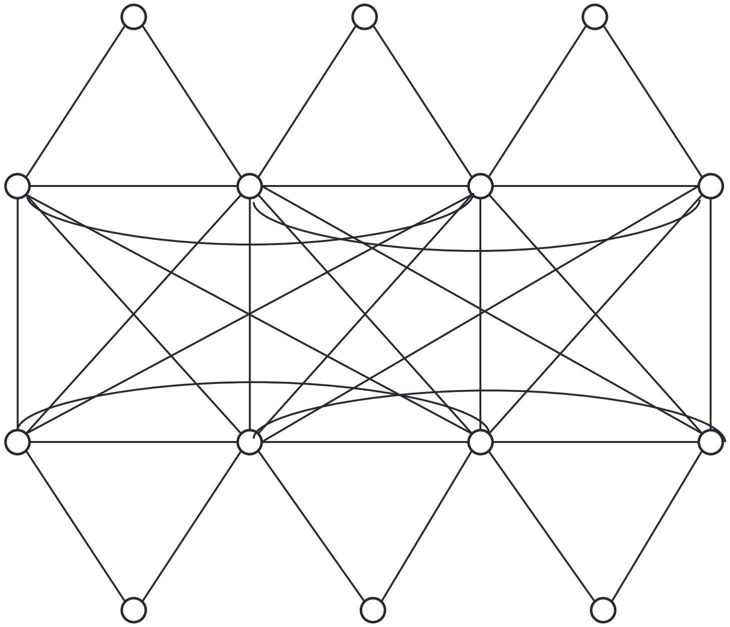
Theorem [L, 2018]

There is an infinity of (minimal) strongly chordal graphs that are not leaf powers.

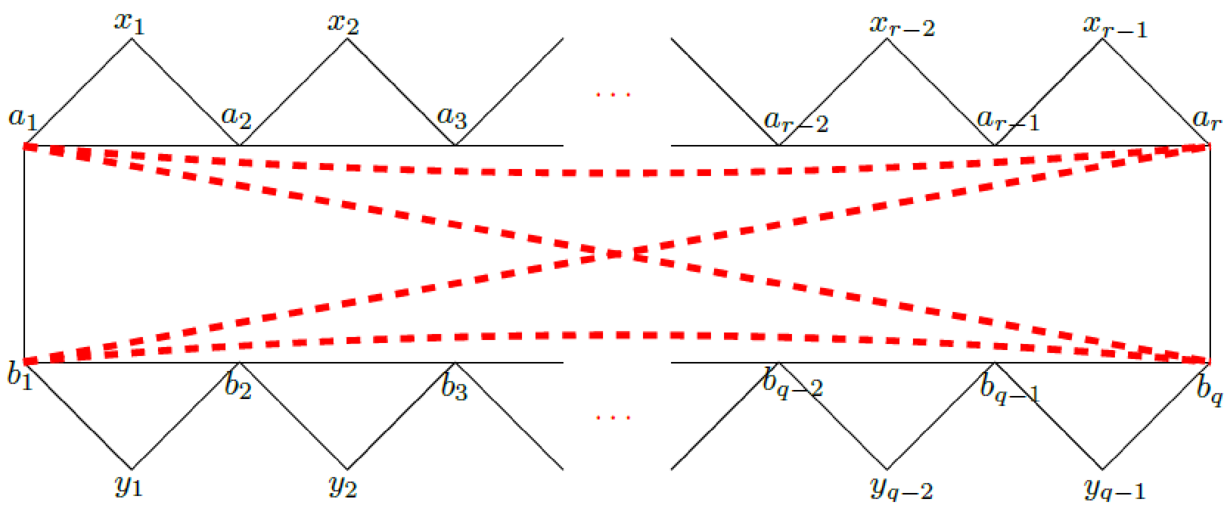
- Here's one of them



- Here's another



- In general: start with paths of length r and q, add spikes on edges, add all edges between paths vertices except four extremities.



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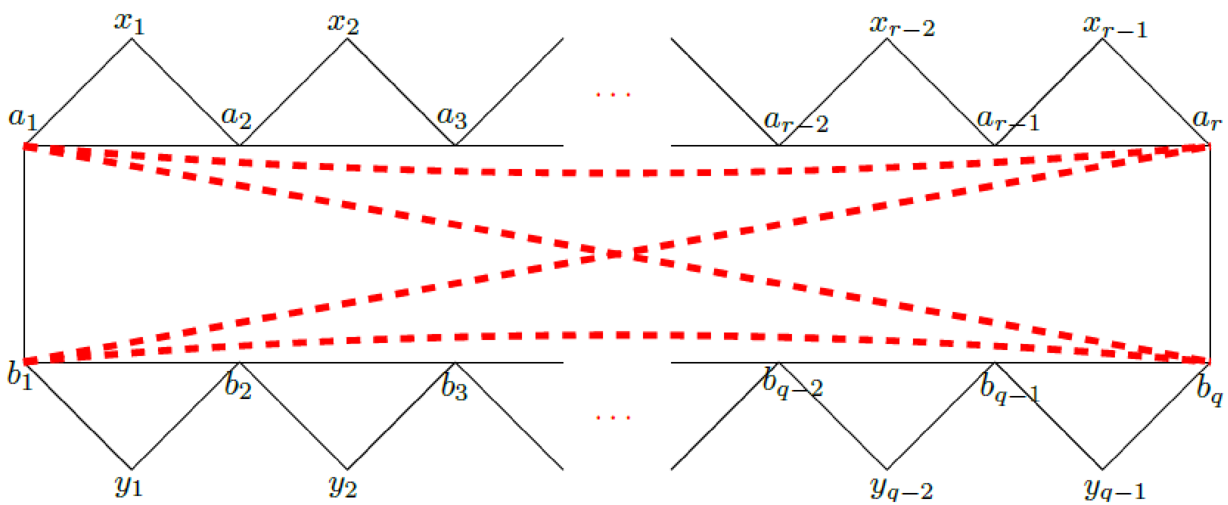
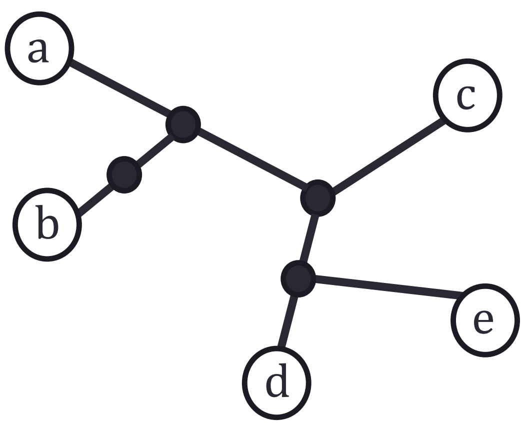


- Not too hard to prove strong chordality.
- But how do we prove non-leaf powerness?

Detecting non-leaf powers from quartets

- Mandatory quartet consistency
  - A new way of identifying non-leaf powers
- **Definition:** a tree T contains a quartet ab|cd if T has an edge that separates {a,b} from {c,d}

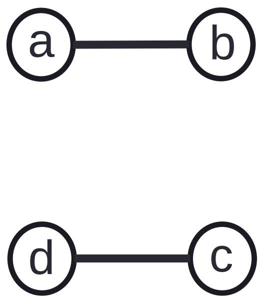


Contains:
ab|cd
ab|ce
ab|de
ac|de
bc|de

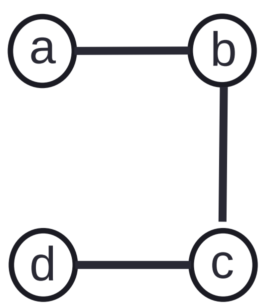
Assume that G is a leaf power explained by some tree T.

If G has one of these...

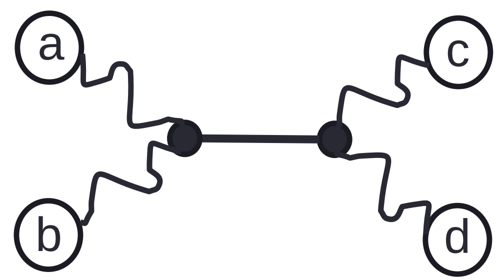
$2K_2$



P_4



..then T must contain the ab|cd quartet.

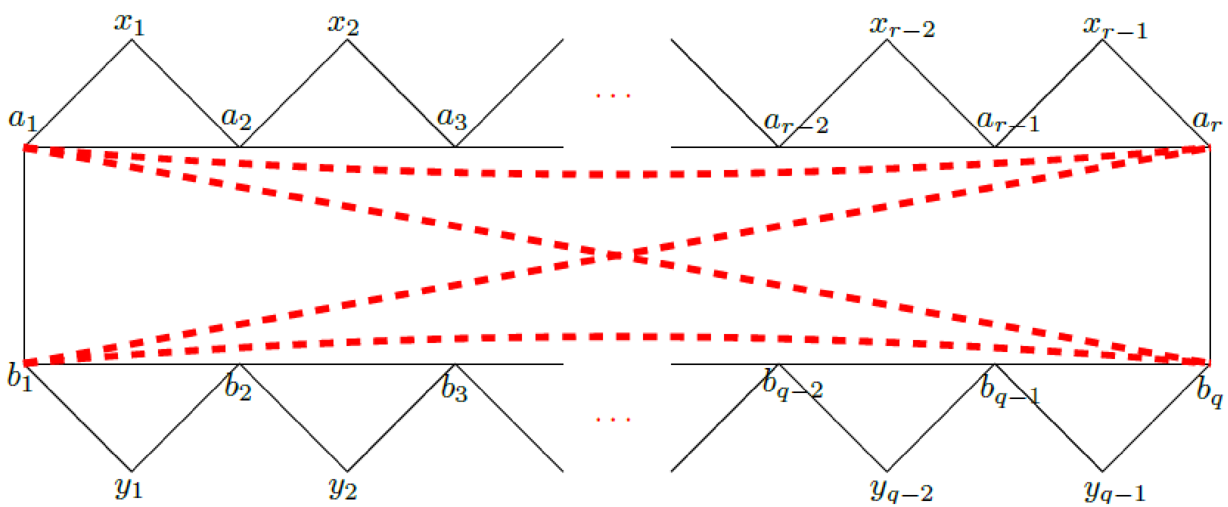


Detecting non-leaf powers from quartets

- List every $2K_2$ and P_4 to infer the mandatory quartets Q.
- If no tree can possibly contain every quartet of Q, then G cannot be a leaf power.

Detecting non-leaf powers from quartets

For these graphs, can be shown to require every $a_i a_{i+1} | b_j b_{j+1}$, and $a_1 b_1 | a_r b_q$ which are known to admit no tree [Shutters et al., 2012]



Some trivia

- NP-hard to decide if a chordal graph contains a copy of some $G_{r,q}$
  - First non-leaf power certificate that we don't know how to check
- Conjecture: can recognize k-leaf powers explained by a f(k) bounded degree tree in poly(n) time.
  - Implications?
- Other necessary conditions based on alternating cycles (edge to non-edge alternation)

Conclusion

- Every question is still open
  - Complexity of recognizing leaf powers and k-leaf powers?
  - Forbidden subgraphs of leaf powers and k-leaf powers?

- Leaf powers and quartets
  - Question: is G a leaf power iff there is a tree that contains all its mandatory quartets?
  - Answer: no, but only one counter-example is known, the 3-sun. All other non-leaf powers have incompatible quartets.